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A Classification of Quantum Hall Fluids

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In this paper, the key ideas of characterizing universality classes of dissipation-free (incompressible) quantum Hall fluids by mathematical objects called quantum Hall lattices are reviewed. Many general theorems about the classification of quantum Hall lattices are stated and their physical implications are discussed. Physically relevant subclasses of quantum Hall lattices are defined and completely classified. The results are carefully compared with experimental data and also with other theoretical schemes (the hierarchy schemes). Several proposals for new experiments are made which could help to settle interesting issues in the theory of the (fractional) quantum Hall effect and thus would lead to a deeper understanding of this remarkable effect.

KEY WORDS: Quantum Hall effect; classification of universality classes of quantum Hall fluids; integral lattices; quadratic forms; lattice embeddings; phase transitions.

1. INTRODUCTION: EXPERIMENTAL FACTS AND THEORETICAL IDEAS

In this paper we describe a classification of (universality classes of) dissipation-free (incompressible) quantum Hall fluids in terms of arithmetic invariants connected to integral lattices. The key insight will be that the theory of certain classes of integral lattices organizes experimental data in an efficient and accurate way. We emphasize that the appearance of integral lattices in the theory of the quantum Hall (QH) effect is not the consequence of queer mathematical fantasizing devoid of physical insight, but is the consequence of some fundamental physical principles and

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properties, such as the absence of dissipation in an incompressible QH fluid, electromagnetic gauge invariance, parity and time-reversal breaking of the quantum mechanics of charged particles in an external magnetic field, and the Fermi statistics of electrons. It is our aim to show that integral lattices are fundamental to the theory of the QH effect. It will therefore be impossible to spare the reader a certain amount of mathematical reasoning involving lattice theory.

The integer QH effect was discovered by von Klitzing and collaborators 15 years ago, the fractional effect by Tsui and collaborators in 1982; see ref. 1. Since then this remarkable effect of non-relativistic many-body physics has posed numerous and diverse challenges to experimentalists and theoreticians. As theorists, we should sadly confess that we have anticipated few of the real surprises.

Experimentally, the QH effect is observed in two-dimensional systems of electrons and/or holes confined to a planar region Ω and under the influence of a strong, uniform magnetic field $\mathbf{B}_c$ transversal to Ω. Such systems can be realized as inversion layers forming at the interface between an insulator and a semiconductor when an electric field (gate voltage) perpendicular to the interface is applied. Imagine that the sample is rectangular, with Ω contained in the (x, y)-plane. By tuning the total electric current $I = (I_x, I_y)$ to some value and measuring the voltage drops V_x and V_y in the x and y directions of the plane of the system, we can determine the resistances R_{xx}, R_{yy}, and R_{H} from the equations

$$\begin{aligned} V_x &= R_{xx} I_x - R_{\mathrm{H}} I_y \\ V_y &= R_{\mathrm{H}} I_x + R_{yy} I_y \end{aligned} \tag{1.1}$$

One finds that at temperatures T very close to 0 K R_{H} is independent of I; it only depends on a dimensionless quantity ν called the *filling factor* and defined by

$$\nu = \frac{n}{eB_c^{\perp}/hc} \tag{1.2}$$

where n is the difference between the density of electrons and the density of holes in the sample, $B_c^{\perp}$ is the component of the external magnetic field $\mathbf{B}_c$ perpendicular to the plane of the sample, and hc/e is the quantum of magnetic flux. Treating electrons and holes as classical point particles, one finds by equating electrostatic and Lorentz force that in a stationary state

$$\frac{1}{R_{\mathrm{H}}} = \nu \frac{e^2}{h} \tag{1.3}$$

The constant of proportionality e^2/h is a universal constant of nature. Since, experimentally, n can be varied (by varying the gate voltage) and $B_c^{\perp}$ can be varied, the classical prediction (1.3) can be tested. Experiments at very low temperatures with rather pure samples yield surprising data: The experimental curve for R_{H}^{-1} as a function of ν shows plateaus, i.e., small intervals of values of ν where R_{H}^{-1} is constant. Whenever $(\nu, R_{\mathrm{H}}^{-1})$ belongs to a plateau, then:

(i) R_{xx} and R_{yy} very nearly vanish.

(ii) R_{H}^{-1} is a *rational* multiple of e^2/h. The plateaus where $R_{\mathrm{H}}^{-1} = n_{\mathrm{H}} e^2/h$ for some integer $n_{\mathrm{H}} = 1, 2, 3, \ldots$ (not too large) occur with an astounding precision of one part in 10^8. The plateau-height quantization is insensitive to sample preparation (e.g., to impurities) and geometry, for all practical purposes.

(iii) Only a limited (experimentally, a finite) set of rational numbers appear as plateauheights of $R_{\mathrm{H}}^{-1} h/e^2$. The behavior of R_{H}^{-1} as a function of ν between neighbouring plateaus appears to exhibit universal features. In such transition regions R_{xx} and R_{yy} are non-zero.

These (and other) experimental findings pose fascinating problems to the theorist:

1. Applying nonrelativistic many-body theory to a two-dimensional system of interacting electrons in an external magnetic field, can one predict the values of ν at which R_{xx} and R_{yy} vanish?

2. If R_{xx} and R_{yy} vanish, can one predict the possible values of R_{H}? Writing

$$R_{\mathrm{H}}^{-1} = \sigma_{\mathrm{H}} \frac{e^2}{h} \qquad \text{with} \quad \sigma_{\mathrm{H}} = \frac{n_{\mathrm{H}}}{d_{\mathrm{H}}} \tag{1.4}$$

where n_{H} and d_{H} are two integers without common divisor, we would like to understand which set of rational numbers $n_{\mathrm{H}}/d_{\mathrm{H}}$ corresponds to plateauheights of the *dimensionless Hall conductivity* (*or Hall fraction*) σ_{H} in real samples. Do only special types of integers appear as numerators n_{H} or denominators d_{H} of σ_{H} ("odd-denominator rule")? Conversely, can we predict which rational numbers will "never" appear as plateauheights of σ_{H}? How does the set of observed plateauheights depend on properties of the sample, e.g., on the number of interacting layers, the width of the quantum well corresponding to a layer, the in-plane component $\mathbf{B}_c^{\parallel}$ of the applied magnetic field, etc.? Given an observed plateauheight of σ_{H}, can we say something about the stability of the corresponding state of the system?

3. What is the structure of the quantummechanical state of the system when (ν, σ_H) lies in between two plateaus: e.g., when $\nu = 1/4$ or $\nu = 1/2$, in a single-layer sample? Experimentally, the transitions between plateaus do not appear to exhibit any hysteresis phenomena. Does this mean that these transitions are continuous and pass through a critical point where one should observe critical phenomena? If this is the case what kind of theories describe the critical points? Can we predict the (relative) widths of plateaus and of transition regions?

During the past 5 years we have been involved in theoretical work on many of these questions. While we feel that theorists have gained a lot of fairly convincing heuristic insight in the direction of answering these questions, it is only the questions described under point 2 above to which we have what we would like to think are fairly definitive and mathematically precise answers. The description and mathematical derivation of some of these answers form the main contents of this paper. (We hope to present some of our insights into questions posed in points 1 and 3 in future communications.)

The ground work for our approach to the problems described under point 2 has been carried out in refs. 2–7. It owes much inspiration to work of Halperin[8] and Read[9] and overlaps with work by Wen and others[10] (see also ref. 1, 11, and 12).

Next we recapitulate the key theoretical facts underlying our analysis. In this work we use units where the electron charge $-e$ and Planck's constant h equal unity. A two-dimensional system of electrons and/or holes in a transversal, external magnetic field exhibiting the Hall effect ($R_H \neq 0$) is called a *QH system*. If R_{xx} and R_{yy} vanish, it is called an *incompressible QH fluid* or, for short, a *QH fluid*.

Our purpose in this paper is to explain or predict *universal* properties of QH fluids at temperatures $T \approx 0$ K. It is therefore reasonable to look for a description of such systems in the scaling limit. Thus we consider a family, parametrized by a scale parameter θ with $1 \leqslant \theta < \infty$, of ever-larger samples confined to regions $\Omega^{(\theta)} := \{\mathbf{x} \mid \mathbf{x}/\theta =: \xi \in \Omega\}$ in the (x, y) plane. We describe the system in $\Omega^{(\theta)}$ in terms of rescaled space and time coordinates (τ, ξ), where $\xi = \mathbf{x}/\theta$, $\mathbf{x} \in \Omega^{(\theta)}$, $\tau = t/\theta$, and $t \in \mathbb{R}$ denotes time. The property that R_{xx} and R_{yy} vanish in QH fluids can be interpreted as indicating that the ground-state energy of such a quantum fluid confined to the region $\Omega^{(\theta)}$ is separated from the rest of its spectrum of energies of (extended) states by a *mobility gap* $\Delta^{(\theta)}$, with

$$\Delta^{(\theta)} \geqslant \Delta_* > 0 \tag{1.5}$$

for all θ. From assumption (1.5) it follows that the universal physics of QH fluids in the scaling limit $\theta \to \infty$ is described by a *topological field theory*.

For the purpose of predicting the values of σ_H or of other electric transport properties, it is sufficient to determine the Green functions of conserved current densities, in particular of the electric current density, in the scaling limit. Thus, let $j_1,..., j_N$ be a list of *all* current densities of a QH fluid which, in the scaling limit, are *independently conserved.* We write

$$j_k(\tau, \boldsymbol{\xi}) = (j_k^0(\tau, \boldsymbol{\xi}), \mathbf{j}_k(\tau, \boldsymbol{\xi})) \tag{1.6}$$

where j_k^0 is the charge density and $\mathbf{j}_k$ the vector current density associated with j_k, $k = 1,..., N$. Saying that j_k is *conserved* means that it satisfies the *continuity equation*

$$\frac{1}{c}\frac{\partial}{\partial \tau} j_k^0 + \nabla \cdot \mathbf{j}_k = 0 \tag{1.7}$$

The *total electric* current density j_{el} must always be among the conserved current densities of a QH fluid. Thus there are real numbers $Q_1,..., Q_N$ such that

$$j_{el} = \sum_{k=1}^{N} Q_k j_k \tag{1.8}$$

Let $\langle \cdots \rangle^{(\theta)}$ denote the quantummechanical expectation in the ground state of a QH fluid confined to $\Omega^{(\theta)}$. Let $\xi := (\xi^0, \xi^1, \xi^2) = (c\tau, \boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Omega$, and $\partial_\mu := \partial/\partial \xi^\mu$. We define the "vacuum polarization tensor" Π in the scaling limit by

$$\Pi_{kl}^{\mu\nu}(\xi, \eta) := \lim_{\theta \to \infty} \theta^4 \langle T[j_k^\mu(\theta\xi)\, j_l^\nu(\theta\eta)]\rangle^{(\theta)} \tag{1.9}$$

for $\mu, \nu = 0, 1, 2$, and $k, l = 1,..., N$. In (1.9) we are using that a conserved current density of a two-dimensional system scales like the square of an inverse length (conserved current densities *cannot* have anomalous scaling dimensions). It follows from the continuity equations (1.7) that

$$\partial_\mu \Pi_{kl}^{\mu\nu} = \partial_\nu \Pi_{kl}^{\mu\nu} = 0 \qquad \text{for all} \quad k, l = 1,..., N \tag{1.10}$$

From (1.10) and the fact that the current densities j_k have scaling dimension 2 it follows that for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in the interior of Ω

$$\Pi_{kl}^{\mu\nu}(\xi, \eta) = iS^{kl}\varepsilon^{\mu\nu\rho}\, \partial_\rho \delta^{(3)}(\xi - \eta) \quad (+ \cdots) \tag{1.11}$$

where the coefficients S^{kl} are the matrix elements of a *symmetric* $N \times N$ matrix S and are *dimensionless* (in our units, where $h = -e = 1$). The terms

$(+\cdots)$ omitted on the r.h.s. of (1.11) involve second or higher derivatives of δ-functions and have *dimensionful* coefficients (with dimensions of a first or higher power of length). They are of subleading order in the scaling limit. Let N_+, N_-, and N_0 denote the number of positive, negative, and zero eigenvalues of S, respectively. By rescaling the current densities j_k and introducing suitable linear combinations thereof, we can always achieve that

$$S^{kl} = s_k \, \delta^{kl} \tag{1.12}$$

with $s_k = 1$ for $1 \leqslant k \leqslant N_+$, $s_k = -1$ for $N_+ + 1 \leqslant k \leqslant N_+ + N_-$, and $s_k = 0$ otherwise. We may henceforth assume that the current densities j_k have been chosen in such a way that (1.12) holds. In discussing electric transport properties *in the scaling limit* and predicting the *possible values* of $\sigma_{\rm H}$, current densities j_k corresponding to $s_k = 0$ are irrelevant, and we may therefore assume that $N_0 = 0$, $N = N_+ + N_-$.

Note that for $S \neq 0$ the tensor Π violates parity and time-reversal invariance. Thus, the ground state of a QH fluid is not invariant under parity and timereversal unless $N_+ = N_- = 0$. This is to be expected of a system of charged particles in an external magnetic field.

It follows from (1.11) and (1.8) that

$$\begin{aligned} \Pi^{\mu\nu}_{\rm el}(\xi, \eta) &:= \lim_{\theta \to \infty} \theta^4 \langle T[j^{\mu}_{\rm el}(\theta\xi) \, j^{\nu}_{\rm el}(\theta\eta)] \rangle^{(\theta)} \\ &= i \langle \mathbf{Q}, \mathbf{Q} \rangle \, \varepsilon^{\mu\nu\rho} \partial_\rho \, \delta^{(3)}(\xi - \eta) \end{aligned} \tag{1.13}$$

where $\mathbf{Q}$ with components $Q_1, ..., Q_N$, introduced in (1.8), is called "*charge vector*," and

$$\langle \mathbf{Q}, \mathbf{Q} \rangle = \sum_{k,\, l=1}^{N} Q_k S^{kl} Q_l = \sum_{k=1}^{N} s_k Q_k^2 \tag{1.14}$$

where the second equality holds if the "normalization conditions" (1.12) are imposed.

From the basic equations of the electrodynamics of QH fluids[2, 5] we know that the coefficient $\langle \mathbf{Q}, \mathbf{Q} \rangle$ on the r.h.s. of (1.13) is nothing but the *dimensionless Hall conductivity* $\sigma_{\rm H}$, i.e.,

$$\sigma_{\rm H} = \langle \mathbf{Q}, \mathbf{Q} \rangle \tag{1.15}$$

Since the theory describing a QH fluid in the scaling limit is a *topological field theory* ($\Delta_* > 0$!), as remarked above, all excitations above the ground state of a QH fluid of finite energy and localized in compact regions contained in the bulk of the system ("*quasi-particles*") can be described, in the scaling limit, as *pointlike*, *static* sources of the topological field theory (located at points in the interior of Ω). One can show[2, 13] that one can assign N charges, $q^1,..., q^N$ to every such source. The charge q^k is an eigenvalue of the conserved total charge operator corresponding to the conserved current density j_k; this charge operator is normalized in such a way that the ground state of the system has charge zero. By (1.8), the total electric charge of a source described by a vector $\mathbf{q}$ of charges $q^1,..., q^N$, is given by

$$q_{\mathrm{el}}(\mathbf{q}) = \sum_{k=1}^{N} Q_k q^k \tag{1.16}$$

If a source with a vector $\mathbf{q}_1$ of charges is transported (adiabatically) around a source with a vector $\mathbf{q}_2$ of charges along a counterclockwise oriented loop not enclosing other sources, a corresponding quantum mechanical state vector is multiplied by an "Aharonov–Bohm phase factor"

$$\exp(2\pi i \langle \mathbf{q}_1, \mathbf{q}_2 \rangle) \tag{1.17}$$

where

$$\langle \mathbf{q}_1, \mathbf{q}_2 \rangle = \sum_{k,l=1}^{N} q_1^k (S^{-1})_{kl} q_2^l \tag{1.18}$$

If two identical sources labeled by vectors $\mathbf{q}_1$, $\mathbf{q}_2$ of charges with $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{q}$ are (adiabatically) *exchanged* along counterclockwise oriented paths not enclosing other sources then a corresponding quantum mechanical state vector is multiplied by the phase factor

$$\exp(\pi i \langle \mathbf{q}, \mathbf{q} \rangle) \tag{1.19}$$

These are properties of physical state vectors of the topological field theory, an abelian Chern–Simons theory of N gauge fields, that reproduces the current Green functions given in (1.11). They have been derived and discussed in great detail in previous papers.[2, 3, 5, 6]

The conventional connection between electric charge and quantum statistics in a quantum mechanical gas of nonrelativistic electrons says that whenever the total electric charge $q_{\mathrm{el}}(\mathbf{q})$ of a localized excitation labeled by a vector $\mathbf{q}$ of charges is an *even* (*odd*) *integer* (in units where $e = -1$), i.e., the excitation is composed of an *even* (*odd*) number of electrons and/or holes, then the excitation obeys *Bose–Einstein* (*Fermi–Dirac*) statistics. This *charge–statistics connection* together with (1.19) implies that every vector $\mathbf{q}$ corresponding to an integer electric charge $q_{\mathrm{el}}(\mathbf{q})$ satisfies the constraint

$$q_{\mathrm{el}}(\mathbf{q}) \equiv \langle \mathbf{q}, \mathbf{q} \rangle \bmod 2 \tag{1.20}$$

Moreover, it follows from the charge–statistics connection and (1.17) that if $\mathbf{q}_1$ and $\mathbf{q}_2$ both correspond to *integer* electric charges $q_{\text{el}}(\mathbf{q}_1), q_{\text{el}}(\mathbf{q}_2) \in \mathbb{Z}$, then $\langle \mathbf{q}_1, \mathbf{q}_2 \rangle$ is an *integer*. Finally, the vectors $\mathbf{q}$ for which $q_{\text{el}}(\mathbf{q})$ is an integer form an *additive group*; addition corresponding to the composition of two excitations, and the operation $\mathbf{q} \to -\mathbf{q}$ corresponds to "charge conjugation" (electron–hole exchange).

A detailed account of the arguments just sketched can be found in ref. 6. The key result that they imply is that the vectors $\mathbf{q}$ of charges belonging to the set

$$\Gamma := \{\mathbf{q} \in \mathbb{R}^N \mid q_{\text{el}}(\mathbf{q}) \in \mathbb{Z},\ q_{\text{el}}(\mathbf{q}) \equiv \langle \mathbf{q}, \mathbf{q} \rangle \bmod 2\} \tag{1.21}$$

form an *integral lattice*. In other words, Γ is an additive group (a "free $\mathbb{Z}$-module") and for any pair $\mathbf{q}_1, \mathbf{q}_2$ of vectors in Γ, $\langle \mathbf{q}_1, \mathbf{q}_2 \rangle$ is an integer. We define the lattice *dual* to Γ by

$$\Gamma^* := \{\mathbf{n} \in \mathbb{R}^N \mid \langle \mathbf{n}, \mathbf{q} \rangle \in \mathbb{Z} \text{ for all } \mathbf{q} \in \Gamma\} \tag{1.22}$$

Since the charge vector $\mathbf{Q}$ introduced in (1.8) and (1.13) has the property that

$$\langle \mathbf{Q}, \mathbf{q} \rangle = q_{\text{el}}(\mathbf{q}) \in \mathbb{Z} \text{ for all } \mathbf{q} \in \Gamma \tag{1.23}$$

it follows that $\mathbf{Q} \in \Gamma^*$. This implies that $\langle \mathbf{Q}, \mathbf{Q} \rangle$ is a rational number and hence, by (1.15), the Hall fraction $\sigma_{\text{H}} = n_{\text{H}}/d_{\text{H}} = \langle \mathbf{Q}, \mathbf{Q} \rangle$ is *rational.*

An electron and a hole are among the localizable, physical excitations of a QH fluid. Thus there must exist some vector $\mathbf{q} \in \Gamma$ with the property that

$$q_{\text{el}}(\mathbf{q}) = \langle \mathbf{Q}, \mathbf{q} \rangle = 1 \tag{1.24}$$

Then (1.20) implies that $\langle \mathbf{q}, \mathbf{q} \rangle$ is an *odd* integer; hence Γ is what is called an *odd* integral lattice, and, by (1.24), $\mathbf{Q}$ is a so-called *primitive* (or *visible*) vector of Γ^*. Moreover, by reading the charge–statistics connection (1.20) (which holds for all $\mathbf{q} \in \Gamma$) as a constraint on $\mathbf{Q}$, we say that $\mathbf{Q}$ is an *odd* vector of Γ^*.

It is a basic fact of nonrelativistic quantum theory that state vectors are single-valued in the positions of electrons and holes. Let $\mathbf{n}$ be a vector of charges of an arbitrary, localizable physical excitation of a QH fluid, and let $\mathbf{q} \in \Gamma$. Then by (1.17) and since state vectors are single-valued in the positions of electrons and holes, $\langle \mathbf{n}, \mathbf{q} \rangle$ must be an integer, and hence

$$\mathbf{n} \in \Gamma^* \tag{1.25}$$

Thus the vectors of charges of localizable physical excitations form a lattice Γ_{phys} contained in or equal to Γ^*.

The conclusion reached, so far, is that: In the scaling limit, an (incompressible) QH fluid with N conserved current densities $j_1,..., j_N$ (we shall speak of N "channels"), $N = 1, 2,...,$ can be characterized by the data: (i) an N-dimensional, odd, integral lattice Γ; (ii) an odd, primitive vector $\mathbf{Q} \in \Gamma^*$ with $\langle \mathbf{Q}, \mathbf{Q} \rangle = \sigma_{\text{H}}$; and (iii) a lattice Γ_{phys} with $\Gamma \subseteq \Gamma_{\text{phys}} \subseteq \Gamma^*$.

A pair $(\Gamma, \mathbf{Q})$ is called a *quantum Hall lattice*. If the integral quadratic form (or metric) $\langle\ ,\ \rangle$ defined on Γ is either *positive-* or *negative-definite*, we say that $(\Gamma, \mathbf{Q})$ is a *chiral* QH lattice (CQHL), for reasons connected to the chirality of edge currents; see Section 2 and also refs. 5 and 6. It is a plausible idea about the physics of QH fluids that if $\langle\ ,\ \rangle$ is *not* positive- or negative-definite, then Γ can be *decomposed* into an (orthogonal) direct sum,

$$\Gamma = \Gamma_e \oplus \Gamma_h \tag{1.26}$$

with the property that $\Gamma_e(\Gamma_h)$ is an odd, integral sublattice of Γ on which $\langle\ ,\ \rangle$ is positive- (negative-) definite. Decomposition (1.26) may not hold in general, but it will serve as a fairly safe "working hypothesis" throughout much of this paper. The physical basis of this working hypothesis (decomposition of QH fluids into electron- and hole-rich subfluids) will be discussed in Section 2 and Appendix E; see also ref. 6 [In Section 2 we summarize the basic physical assumptions of our approach and provide the mathematical notions connected to (chiral) QH lattices.]

Our aim in this paper is to present a *partial classification* of QH lattices. In view of our working hypothesis (1.26), our main effort will concern the classification of *chiral* QH lattices (but see Appendix E). We shall carefully compare our results with experimental data on QH fluids, focusing our attention primarily on data for single-layer QH fluids with σ_{H} in the interval $0 < \sigma_{\text{H}} \leqslant 1$. Our job involves a characterization of QH lattices $(\Gamma, \mathbf{Q})$ in terms of numerical invariants; see Section 3. Among these invariants, the following ones play a key role:

(i) The dimension N of Γ.

(ii) The discriminant of Γ, i.e., the order of the Abelian group Γ^*/Γ, where Γ^*/Γ denotes the family of cosets of Γ^* mod Γ (as well as more sophisticated invariants involving Γ^*/Γ, e.g., the genus of Γ).

(iii) An invariant, denoted $l_{\max}$, interpreted physically as the smallest relative angular momentum of a certain pair of two identical excitations of electric charge 1 (electrons)—$l_{\max}$ is an *odd* integer (see Section 3).

(iv) And of course the dimensionless Hall conductivity (or Hall fraction), $\sigma_{\mathrm{H}} = \langle \mathbf{Q}, \mathbf{Q} \rangle$.

For CQHLs the invariants $l_{\max}$ and σ_{H} are related by

$$l_{\max} \geqslant 1/\sigma_{\mathrm{H}} \tag{1.27}$$

which is a consequence of the Cauchy–Schwarz inequality; see Section 4.

In our comparison between theory and experiment, we shall appeal to a heuristic (analytically plausible, but mathematically unproven) *stability principle* which says that a QH fluid described by a QH lattice $(\Gamma, \mathbf{Q})$ is the more *stable*, the *smaller* the value of the invariant $l_{\max}$ and, given the value of $l_{\max}$, the *smaller* the dimension N (and the discriminant) of Γ; see Sections 4, 6, and 7. A measure for the stability of a QH fluid is, for example, the width of the plateau of σ_{H} (as a function of ν) corresponding to that QH fluid.

In view of (1.27) it is useful to decompose the interval $(0, 1]$ of values of σ_{H} into subintervals ("windows") $\Sigma_p = \Sigma_p^+ \cup \Sigma_p^-$, where

$$\Sigma_p^+ := \left[\frac{1}{2p+1}, \frac{1}{2p}\right), \qquad \Sigma_p^- := \left[\frac{1}{2p}, \frac{1}{2p-1}\right), \qquad p = 1, 2,... \tag{1.28}$$

The invariant $l_{\max}$ of a CQHL $(\Gamma, \mathbf{Q})$ with $\sigma_{\mathrm{H}} \in \Sigma_p$ is bounded below by $2p+1$. We define $\mathscr{H}_p^{\pm}$ to be the class of all CQHLs, $(\Gamma, \mathbf{Q})$, with $\sigma_{\mathrm{H}} \in \Sigma_p^{\pm}$ and $l_{\max} = 2p+1$ (and which are, to be technically precise, "primitive," as specified in Section 2). We shall see that for all p, *all* CQHLs in $\mathscr{H}_p^+$ can be enumerated explicitly, and that for $p \leqslant 3$ and sufficiently small values of their dimension (stability principle!) they correspond to experimentally well verified plateaus of σ_{H}.

There are heuristic analytical and numerical arguments, as well as convincing phenomenological evidence, indicating that the most stable state of a QH system with $\nu < 1/7$ is one where the electrons form a *Wigner lattice*. But a Wigner lattice is incompatible with a positive mobility gap Δ_*, i.e., with incompressibility. By (1.27), this implies that the invariant $l_{\max}$ of a *chiral* QH lattice corresponding to an experimentally realizable QH fluid is bounded above by

$$l_{\max} \leqslant 7 \tag{1.29}$$

There is reasonable, analytical evidence[14] that *single-layer* QH systems with filling factors $\nu = 1/2, 1/4$, and various other even-denominator fractions are described by gapless (possibly marginal) Fermi liquids. Thus, e.g., $\sigma_{\mathrm{H}} = 1/2$ and $\sigma_{\mathrm{H}} = 1/4$ should *not* correspond to plateaus of *single-layer* QH fluids.

The CQHL ($\Gamma = \mathbb{Z}, \mathbf{Q} = 1$) has invariants $N = 1$, $|\Gamma^*/\Gamma| = 1$, $l_{\max} = 1$, and $\sigma_{\rm H} = 1$. It describes the *by far most stable* QH fluid with a Hall conductivity $\sigma_{\rm H} \in (0, 1]$. Thus the plateau at $\sigma_{\rm H} = 1$ should have *by far the broadest width* among *all* plateaus at values of $\sigma_{\rm H}$ in the interval $(0, 1]$. QH fluids described by QH lattices of dimension $N > 1$, discriminant > 1, and with $\sigma_{\rm H}$ close to 1 (e.g., $6/7 \lesssim \sigma_{\rm H} < 1$) are expected to be very unstable against transitions to the QH fluid at $\sigma_{\rm H} = 1$ described by $(\Gamma, \mathbf{Q}) = (\mathbb{Z}, 1)$ and are therefore likely to be *invisible* experimentally.

In Fig. 1 we display experimentally observed plateau values of $\sigma_{\rm H}$ in the interval $0 < \sigma_{\rm H} \leqslant 1$ and indicate the quality of their experimental verification. [For general experimental reviews of the (fractional) QH effect, see, e.g., refs. 15 and 16 and references therein. Recent data on QH fluids with Hall fractions belonging to the "two main series" $\sigma_{\rm H} = N/(2N \pm 1)$, $N = 1, \ldots, 9$, can be found in refs. 17 and 18. For the status of a QH fluid with $\sigma_{\rm H} = 10/17$, see ref. 19. The signals observed at $\sigma_{\rm H} = 4/11$ and $\sigma_{\rm H} = 4/13$ appear to be very weak.[20] Magnetic field and density-driven phase transitions have been reported at $\sigma_{\rm H} = 2/3$.[21–23] A magnetic field-driven phase transition at $\sigma_{\rm H} = 3/5$ has been established[23] and a possible phase transition at $\sigma_{\rm H} = 5/7$ has been discussed.[16]] In Fig. 1 we write $\sigma_{\rm H} = n_{\rm H}/d_{\rm H}$ and display the data in a "$d_{\rm H}$ versus $\sigma_{\rm H}$ plot." We subdivide the interval $(0, 1]$ into the windows $\Sigma_p^{\pm}$ introduced above for $p = 1, 2$, and 3.

It may happen that there are *several* QH lattices with the *same* Hall fraction $\sigma_{\rm H}$. At such values of $\sigma_{\rm H}$ we predict *phase transitions between "structurally different" QH fluids*, as, e.g., the in-plane component $\mathbf{B}_c^{\parallel}$ of the external magnetic field (and thus the magnitude of Zeeman energies associated with the magnetic moment of electrons), or the density of electrons (at fixed filling factor), or the width of the layer to which the electrons (or holes) are confined are varied. A theory of such phase transitions is developed in Section 7 and the results are summarized in Appendix D. The most likely Hall fractions $\sigma_{\rm H}$ at which they may occur are 2/3, 3/5, 4/7, 5/7, 5/9, and 1/2!

We shall find (see Section 5 and Appendix B) a nice, simple CQHL $(\Gamma, \mathbf{Q})$ with $N = 3$, $l_{\max} = 3$ and $\sigma_{\rm H} = 1/2$. However, in *single-layer* QH systems, there is *no* plateau at $\sigma_{\rm H} = 1/2$, and we just said that there is analytical evidence for the idea that the ground state of a QH system at $\nu = 1/2$ is a gapless Fermi liquid. So is there a problem with our theory? In order to understand what is going on at $\sigma_{\rm H} = 1/2$ (and at various other values of $\sigma_{\rm H} \in (0, 1]$), it is useful to consider yet one further invariant of integral lattices, the so-called *Witt sublattice*. Given an integral lattice Γ, its Witt sublattice $\Gamma_{\rm W}$ is defined to be the sublattice generated by all vectors $\mathbf{q} \in \Gamma$ with $\langle \mathbf{q}, \mathbf{q} \rangle = 1$ or 2. It turns out that, for (*indecomposable*) *chiral*

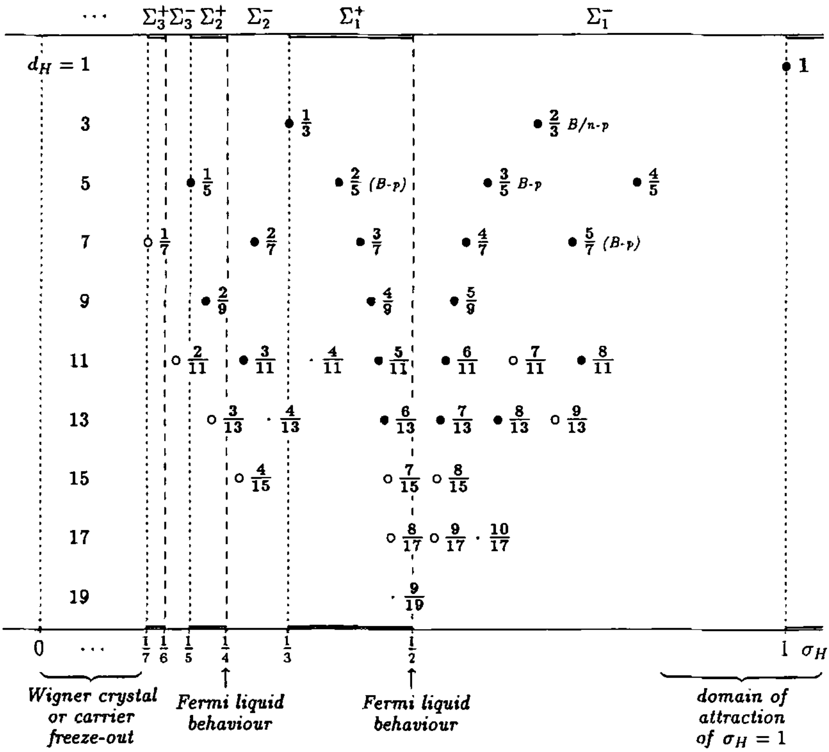

Fig. 1. Observed Hall fractions $\sigma_H = n_N/d_N$ in the interval $0 < \sigma_H \leqslant 1$ and their experimental status in single-layer quantum Hall systems. (•) Well-established Hall fractions; these are fractions for which an R_{xx} minimum and a plateau in R_H have been clearly observed, and the quantization accuracy of $\sigma_H = 1/R_H$ is typically better than 0.5%. (○) Fractions for which a minimum in R_{xx} and typically an inflection in R_H (i.e., a minimum in $dR_H/dB_c^\perp$, but no well-developed plateau in R_H) have been observed. If there are only very weak experimental indications or controversial data for a given Hall fraction, the symbol (•). Finally, $B/n-p$ is appended to fractions at which a magnetic field (B)-and/or density (n)- driven phase transition has been observed.

QH lattices $(\Gamma, \mathbf{Q})$, the Witt sublattice Γ_W of Γ is always the *root lattice* of a semisimple Lie algebra $\mathscr{G}$; more precisely, Γ_W is an orthogonal direct sum of A-, D-, and $E_{6,7}$-root lattices. (These notions are explained in Appendix A.) Furthermore, the Lie group $\mathscr{G}$ corresponding to the Lie algebra $\mathscr{G}$ whose root lattice is given by Γ_W is a *symmetry group* of the topological quantum theory describing the scaling limit of the QH fluid corresponding to $(\Gamma, \mathbf{Q})$, in a sense that has been made precise in refs. 3, 5, and 6 and is briefly reviewed in Section 5. Standard physics often permits us to determine at least some of the symmetries of QH fluids (in the scaling limit).

For example, if the *effective* gyromagnetic factor of an electron in a QH fluid is *small*, so that Zeeman energies can be essentially neglected, then the scaling limit of a QH fluid in an only moderately large magnetic field is expected to exhibit an $SU(2)_{\text{spin}}$ global symmetry (spin flip).[24, 5] In this case the Witt sublattice Γ_{W} of the QH lattice describing the QH fluid must contain the root lattice $\sqrt{2}\,\mathbb{Z}$ of $su(2)$. Furthermore, if we consider a *double-layer* QH fluid, which, in the scaling limit, exhibits an $SU(2)_{\text{layer}}$ symmetry [coherent superposition of modes in the two layers with $SU(2)$ symmetry] then Γ_{W} must contain an $su(2)$-root lattice. One can easily imagine that there are double-layer QH fluids exhibiting (in the scaling limit) both symmetries, an $SU(2)_{\text{spin}}$ and an $SU(2)_{\text{layer}}$ symmetry. Then Γ_{W} must contain the direct sum of *two* $su(2)$-root lattices. It so happens that there is a *three-dimensional CQHL* $(\Gamma, \mathbf{Q})$ with $l_{\max} = 3$, $\Gamma_{\text{W}} = \sqrt{2}\,\mathbb{Z} \oplus \sqrt{2}\,\mathbb{Z}$ [direct sum of two $su(2)$-root lattices], and $\sigma_{\text{H}} = 1/2$. This matches the recent experimental observation of a plateau at $\sigma_{\text{H}} = 1/2$ in double-layer (or two-component) QH systems.[25, 26]

Incidentally, "layer" could also stand for "filled Landau level" and this remark suggests a theoretical explanation of the observed plateau at $\sigma_{\text{H}} = 5/2$.[27, 28]

There is also a *two-dimensional* CQHL $(\Gamma, \mathbf{Q})$ with $l_{\max} = 3$, $\Gamma_{\text{W}} = \varnothing$, and $\sigma_{\text{H}} = 1/2$. It might describe an incompressible QH fluid consisting of *two* interacting layers of *spin-polarized* electrons with a $\mathbb{Z}_2$ layer permutation symmetry. Since $\mathbb{Z}_2$ is a discrete symmetry, it does not contribute to Γ_{W}, but constrains the structure of $(\Gamma, \mathbf{Q})$.

The moral to be drawn from this discussion is that we are well advised to search for global symmetries (discrete and, especially, continuous ones) of the theory that describes the scaling limit of a QH fluid. The continuous symmetries appear as root lattices contained in the Witt sublattice of the QH lattice describing the fluid.

It has been shown in ref. 6 that for CQHLs $(\Gamma, \mathbf{Q})$ with $\sigma_{\text{H}} < 2$,

$$\langle \mathbf{Q}, \mathbf{q} \rangle = 0 \qquad \text{for all} \quad \mathbf{q} \in \Gamma_{\text{W}} \tag{1.30}$$

i.e., $\mathbf{Q}$ is orthogonal to Γ_{W}. Let Γ_0 denote the sublattice of Γ consisting of *all* vectors in Γ that are orthogonal to $\mathbf{Q}$. Clearly, for $\sigma_{\text{H}} < 2$, Γ_0 contains Γ_{W} and obviously $\dim \Gamma_0 \leqslant \dim \Gamma - 1$.

These remarks suggest that an interesting class of QH lattices consists of those CQHLs $(\Gamma, \mathbf{Q})$ for which

$$\Gamma_0 = \Gamma_{\text{W}} \qquad \text{and} \qquad \dim \Gamma_{\text{W}} = \dim \Gamma - 1 \tag{1.31}$$

We call such lattices "*maximally symmetric*" CQHLs. Section 5 is devoted to a classification of all maximally symmetric CQHLs with $0 < \sigma_{\text{H}} \leqslant 1$.

Recall that $\mathscr{H}_p^{\pm}$, $p=1, 2,...$, has been defined to be the class of all (primitive) CQHLs with $\sigma_{\mathrm{H}} \in \Sigma_p^{\pm}$ [see (1.28)] and with $l_{\max}=2p+1$ [which, by (1.27), is the *minimal* value the invariant $l_{\max}$ can have, for $\sigma_{\mathrm{H}} \in \Sigma_p$]. Lattices in $\mathscr{H}_p^{\pm}$ are said to be *L-minimal*. We shall show that *all* lattices in $\mathscr{H}_p^{+}$ are *maximally symmetric*, their Witt sublattice is an A_{N-1}- [or $su(N)$-] root lattice, and their Hall fraction is $\sigma_{\mathrm{H}}=N/(2pN+1)$, $N=(1), 2, 3,...$; see Section 4. This series of CQHLs is called the "basic" A- [or $su(N)$-] series in the window Σ_p. We shall find a *bijection* $\mathscr{S}_p$, called a *shift map*, mapping the basic A-series in the window Σ_1^+ onto the basic A-series in the window Σ_{p+1}^+, $p=1, 2,...$. In fact, the shift map $\mathscr{S}_p$ is defined on $\mathscr{H}_q^{\pm}$ and is a bijection from $\mathscr{H}_q^{\pm}$ to $\mathscr{H}_{q+p}^{\pm}$. On the sets $\mathscr{H}_q^{\pm}$ the action of the map $\mathscr{S}_p$ on the invariants σ_{H} and $l_{\max}$ is given by

$$\frac{1}{\sigma_{\mathrm{H}}} \to \frac{1}{\sigma_{\mathrm{H}}}+2p \qquad \text{and} \qquad l_{\max} \to l_{\max}+2p, \qquad p=1, 2,... \tag{1.32}$$

If $(\Gamma', \mathbf{Q}')$ is the image of a CQHL $(\Gamma, \mathbf{Q})$ under $\mathscr{S}_p$, $p=1, 2,...$, then by (1.32), and invoking our stability principle, the QH fluid corresponding to $(\Gamma', \mathbf{Q}')$ is *less stable* then the one corresponding to $(\Gamma, \mathbf{Q})$. Hence the number of observed plateau values in a window Σ_p *decreases* with p (reaching 0 when $p>3$).

The existence of the shift maps $\mathscr{S}_p$ and the observation just described allow us to restrict our classification of L-minimal CQHLs to the class $\mathscr{H}_1=\mathscr{H}_1^+ \cup \mathscr{H}_1^-$. This is *not* true if we want to classify *all* QH lattices, not just chiral ones. However, among QH lattices that are *not* chiral, the "non-euclidean hierarchy lattices" are well understood (see Appendix E) and they are perhaps the only physically important nonchiral QH lattices. *All* CQHLs in $\mathscr{H}_1^+$ are classified and are maximally symmetric, as remarked above and proven in Section 5. They form the basic A-series in Σ_1^+. The classification of lattices in $\mathscr{H}_1^-$ is much more difficult and remains incomplete. But besides the maximally symmetric ones (Section 5), we have also classified *all* CQHLs in $\mathscr{H}_1^-$ of dimension $N \leqslant 4$. Our results can be found in Section 6. (With more investment in programming and computer time, our results could be extended to $N=5, 6$.)

In Fig. 2, results of our theoretical work concerning QH lattices with odd-denominator Hall fraction are superposed on the experimental data (displayed in Fig. 1) in the window $\Sigma_1=\Sigma_1^+ \cup \Sigma_1^-$. This figure shows a pretty remarkable agreement between theory and experiment. *All* experimentally observed Hall fractions σ_{H} in the window Σ_1, with the only exception of the "very weak" fraction $\sigma_{\mathrm{H}}=4/11$, can be realized by an L-minimal CQHL or a QH lattice which is "charge-conjugated" to an

L-minimal one. Note that the corresponding lattices are all of relatively low dimension, namely $N \leqslant 9$. In Section 6 we shall see that, interestingly, the "simplest" *non-L*-minimal CQHL is found at $\sigma_H = 4/11$. (It coincides with the proposals of the hierarchy schemes at that fraction.) Figure 2 also shows where experimentalists might wish to look for signals of new QH fluids, or for new phase transitions between structurally different QH fluids with the same value of σ_H.

Meditating on Fig. 2, it may look disturbing that one seems to have observed a phase transition at $\sigma_H = 2/5$ as the in-plane component $\mathbf{B}_c^{\parallel}$ of the external magnetic field is varied. There is a *unique* L-minimal CQHL with $\sigma_H = 2/5$. It is two-dimensional, with $\Gamma_W = \sqrt{2}\mathbb{Z}$ [the root lattice of $su(2)$]. So a QH fluid with $\sigma_H = 2/5$ exhibits a global $SU(2)$ symmetry (in the scaling limit). For "*small*" values of the external magnetic field $\mathbf{B}_c$ this symmetry is $SU(2)_{\text{spin}}$, i.e., electron spins may be flipped. But when $\mathbf{B}_c$ is *large* essentially all electron spins are oriented in the direction antiparallel to $\mathbf{B}_c$ and the $SU(2)$ symmetry is an *internal* symmetry compatible with the hierarchy pictures of refs. 29 and 30. A rather similar story can be told about the plateau at $\sigma_H = 2/3$, (besides the possibilities of interesting phase transitions between structurally different QH fluids). All this and more is discussed in Section 7.

Two concluding remarks may be clarifying:

(i) The term "incompressible QH fluid" can be understood literally in that shape fluctuations of a droplet of an (incompressible) QH fluid with free boundaries are *area-preserving*. The Lie algebra of area-preserving maps has a central extension which is connected to the $W_{1+\infty}$ algebra. This algebra is related to the Abelian Chern–Simons theory that describes the scaling limit of an (incompressible) QH fluid in a natural way first discussed by Sakita.[32] Its study in connection with the QH effect has become a "hot topic" (see, e.g., ref. 33), but does not appear to lead to results that go beyond those in refs. 5 and 6, and in this paper.

(ii) The shift map $\mathscr{S}_1: \sigma_H^{-1} \to \sigma_H^{-1} + 2$ and the map $T: \sigma_H \to \sigma_H + 1$, corresponding to the addition of a full Landau level, generate a subgroup $\Gamma_T(2)$ of the modular group $PSL(2, \mathbb{Z})$. For fun, one can study the action of $\Gamma_T(2)$ on the plateau values of σ_H. More daringly, one can study the action of $\Gamma_T(2)$ on the complex plane of resisitivites $\rho := \rho_{xx} + i\sigma_H^{-1}$ (where $\rho_{xx} := R_{xx} l_y / l_x$, with l_x and l_y the length and width, respectively, of a rectangular QH system). This has been advocated in ref. 34 as a means to understand a "global phase diagram" for the QH effect. However, the reader who will make it through Section 4 will see that these are rather misleading speculations which, in the absence of real understanding of the physics of QH systems, should not be taken too seriously.

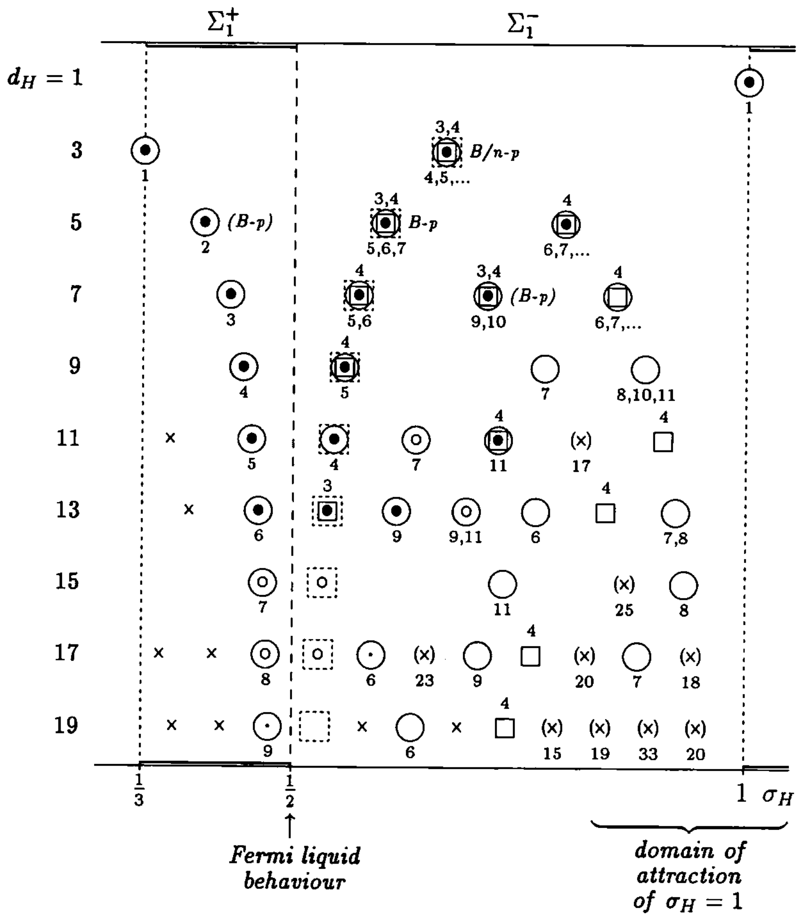

Fig. 2. Compilation of L-minimal ($l_{max} = 3$) chiral quantum Hall lattices (CQHLs) with odd-denominator Hall fractions σ_H in the interval $1/3 < \sigma_H \leqslant 1$. The experimental status of the Hall fractions displayed here is indicated, for single-layer systems by •, ◦, and · as in Fig. 1. Superposed on the interval $1/3 \leqslant \sigma_H \leqslant 1$ of Fig. 2 is a list of different L-minimal CQHLs: ○, maximally symmetric, L-minimal CQHLs of dimension $N \leqslant 11$ (where the corresponding dimensions are given below the symbols); □, generic, indecomposable, L-minimal CQHLs of low dimension, $N \leqslant 4$ (the respective dimensions are given above the symbols). (×), fractions for which there are no low-dimensional ($N \leqslant 4$), L-minimal CQHLs, but there are maximally symmetric ones in "high" dimensions (with the lowest such dimension indicated below the symbols). ×, fractions for which there are neither low-dimensional, L-minimal CQHLs, nor maximally symmetric ones in "higher" dimensions. Dashed boxes stand for nonchiral QH lattices that are "charge-conjugated" to the maximally symmetric, L-minimal CQHLs in Σ_1^+.

As to the contents of this paper, we have already indicated the contents of Sections 5–7. In Section 2 we recall the basic (physical and mathematical) notions underlying our analysis. In Section 3, we introduce and discuss basic invariants for CQHLs and explain their physical interpretation. In Section 4 we present *general* results on the classification of CQHLs. Sections 5 and 6 concern the complete classification of *special subclasses* of CQHLs. In Section 7 we apply our results to understand some of the *physics* of the fractional QH effect. In Appendix A we review some group theory that is important in our analysis. Appendix B summarizes all our results on *maximally symmetric* CQHLs with $\sigma_H \in (0, 1]$, Appendix C those on *low-dimensional* ($N \leqslant 4$) CQHLs. In Appendix D we summarize the results of the theory of embeddings (expounded in Section 7) of L-minimal CQHLs into bigger ones, preserving the value of the Hall fraction σ_H. This will clarify the classification of the "difficult" classes $\mathscr{H}_p^-$. Finally, in Appendix E we present the QH lattices that reproduce the Haldane–Halperin[29] and Jain–Goldman[30] hierarchy fluids.

2. UNIVERSALITY CLASSES OF QH FLUIDS AND QH LATTICES: BASIC NOTIONS

In this section, we recall the basic physical principles and assumptions leading to our characterization of (universality classes of) QH fluids by QH lattices. We introduce the fundamental mathematical notion of a chiral QH lattice (CQHL). As mentioned in Section 1, CQHLs are the "basic building blocks" of QH lattices. Basic notions related to CQHLs are summarized. In order to exemplify our language we describe the (chiral) QH lattices corresponding to the integer QH fluids of the noninteracting electron approximation and the celebrated Laughlin fluids.[35]

Since the early theoretical work by Laughlin[35] on the QH effect the electromagnetic gauge symmetry of quantum mechanics has been instrumental in analyzing this effect. This gauge symmetry also provides the cornerstone of our approach.[2–6] We remark that a general framework for a systematic discussion of phenomena related to electron spin in QH fluids has been developed in refs. 4 and 5. It is based on the presence of a non-Abelian $SU(2)_{\text{spin}}$-gauge symmetry in nonrelativistic quantum many-body systems. Although we will not review that general framework here, we emphasize that our results presented in this paper are fully consistent with that general picture, and, as a matter of fact, the present classification results provide a basis for a systematic discussion of spin effects in QH fluids. For further discussion of this point, see the remarks about phase transitions in Section 7 and refs. 5 and 6. Besides gauge invariance, our approach requires the following basic physical assumptions characterizing QH fluids (see also Section 1):

(A1) The temperature T of the system is close to 0 K. For an (incompressible) QH fluid at $T = 0$ K, the *total electric charge* is a good quantum number to label physical states of the system describing excitations above the ground state.[6, 13] The charge of the ground state of the system is normalized to be zero.

(A2) In the regime of very short wave vectors and low frequencies, the *scaling limit*, the total electric current density is the sum of $N = 1, 2, 3,...$ *separately conserved* $u(1)$-current densities describing electron and/or hole transport in N separate "channels" distinguished by conserved quantum numbers. In our analysis, we regard N as a free parameter. Physically, N turns out to depend on the filling factor ν and other parameters characterizing the system.

(A3) In our units where $h = -e = 1$ the electric charge of an *electron/hole* is $+1/-1$. Any local excitation (quasiparticle) above the ground state of the system with *integer* total electric charge q_{el} satisfies *Fermi–Dirac statistics* if q_{el} is *odd* and *Bose–Einstein statistics* if q_{el} is *even*.

(A4) The quantum-mechanical state vector describing an *arbitrary* physical state of an (incompressible) QH fluid is *single-valued* in the position coordinates of all those (local) excitations that are composed of *electrons* and/or *holes*.

In addition to these four basic assumptions, we put forward, as in refs. 4–7 a "working hypothesis" expressing a "*chiral factorization*" property of QH fluids.

(A5) The fundamental charge carriers of a QH fluid are electrons and/or holes. We assume that in the scaling limit the dynamics of electron-rich subfluids of a QH fluid is *independent* of the dynamics of hole-rich subfluids, and, up to charge conjugation, the theoretical analysis of an electron-rich subfluid is identical to that of a hole-rich subfluid.

We make a few remarks on these assumptions. For a finite, but macroscopic system, assumption (A2) implies that there are N distinct, approximately conserved chiral edge currents circulating along the boundary of the system.—Strict conservation of these $u(1)$-current densities holds in the scaling limit.—This generalizes to the fractional QH effect Halperin's edge current picture[8] of the integer QH effect.[2, 5, 11, 12] Assumption (A5) implies that for an electron-rich QH fluid, say, the chirality of *all* edge currents is the *same*. It is fixed by the direction of the external magnetic field. The mathematical virtue of the edge current picture is that it allows for a natural introduction of the tools of current algebra into the theory of the QH effect; see refs. 10, 2 and 5 and references therein. A systematic mathematical implementation of assumptions (A1)–(A5) in the edge current picture of the QH effect is given in refs. 5 and 6.

Given the close relationship between two-dimensional chiral conformal field theory and quantum Chern–Simons theory, as expounded first by Witten,[36] one can establish a boundary–bulk duality in QH fluids. By this duality, quasi-particles excited at the edge of a QH fluid have their precise counterparts in local bulk sources in a quantum Chern–Simons theory that is expressed in terms of the "vector potentials" of N separately conserved $u(1)$-current densities of the system. Details of this bulk picture and in particular the explicit implementation of assumptions (A1)–(A5) in this picture are given in refs. 6 and 7. Further considerations elucidating the boundary–bulk duality in QH fluids can be found in refs. 5 and 37.

As recapitulated in Section 1, it follows from assumptions (A1)–(A4) that the properties of a QH fluid in the scaling limit can be described completely in terms of a mathematical object that we have called a *quantum Hall lattice*. A QH lattice $(\Gamma, \mathbf{Q})$ consists of an odd, integral lattice Γ and an integer-valued linear functional $\mathbf{Q}$ on Γ; see Section 1 and the definitions below. The number of positive eigenvalues of the metric on Γ corresponds physically to the number of edge currents of one chirality, the number of negative eigenvalues to the number of edge currents of the opposite chirality. In the situation envisaged in the working hypothesis (A5), Γ is an orthogonal direct sum of a lattice Γ_e on which the metric is positive-definite and a lattice Γ_h on which it is negative-definite; see (1.26). The structure of Γ can hence be understood if we are able to enumerate *positive-definite* lattices. In the most general situation, however, Γ could be an *indecomposable*, indefinite lattice or contain an indecomposable, indefinite sublattice. In this case, there would exist local physical excitations of the system of edge currents with the quantum numbers of the electron (electric charge 1 and Fermi–Dirac statistics) that are composed of left- *and* right-moving excitations which themselves, however, are *not* physical quasiparticles of the system. In other words, the left- and right-moving channels of edge currents are coupled to each other in such a way that physical states on the algebra of edge currents *cannot* be factorized into a product of a physical state on the algebra of left-moving edge currents and a physical state on the algebra of right-moving edge currents. We believe that those indecomposable, indefinite lattices do not correspond to stable QH fluids.

While we have not found *a priori* reasons to rule out indecomposable, indefinite (sub)lattices Γ, we shall not consider this situation in the present paper. Rather, it is our strategy to adopt the chirality assumption (A5) as a working hypothesis, and, investigating its strongly predictive consequences, we intend to lay the ground for testing it in different experimental situations; for the predictions, see Fig. 2 in Section 1 and the discussion in Section 7.

In this context we note that all the Haldane–Halperin[29] and Jain–Goldman[30] "hierarchy fluids" satisfy our assumptions (A1)–(A4) and most of them satisfy assumption (A5), too. The exceptions (corresponding to *non-Euclidean*, composite QH lattices) can be shown to satisfy a slightly weaker form of (A5). This slightly weaker form of (A5) is given in Appendix E, where details about "hierarchy QH lattices" can be found.

The stronger assumption (A5) helps in reducing the classification problem of QH fluids to a tractable one! Furthermore, it leads, as we wish to show in this paper, to interesting results typically complementing and sometimes challenging the commonly accepted hierarchy schemes of the QH effect.

Defining an (incompressible) QH fluid as a two-dimensional electronic system with vanishing resistances R_{xx} and R_{yy} [see (1.1)] and satisfying assumptions (A1)–(A5), we advanced the following contention in refs. 5–7.

Classification of QH Fluids. In the scaling limit, the quantum-mechanical description of an (incompressible) QH fluid is universal and completely coded into a pair of chiral quantum Hall lattices (CQHLs), one CQHL $(\Gamma_e, \mathbf{Q}_e)$ for the electron-rich subfluids and one $(\Gamma_h, \mathbf{Q}_h)$ for the hole-rich subfluids.

In our units where $e^2/h = 1$, the Hall conductivity of the entire QH fluid is given by

$$\sigma_{\mathrm{H}} = \langle \mathbf{Q}_e, \mathbf{Q}_e \rangle - \langle \mathbf{Q}_h, \mathbf{Q}_h \rangle = \sigma_{\mathrm{H}}^e - \sigma_{\mathrm{H}}^h \tag{2.1}$$

where $\langle \mathbf{Q}_e, \mathbf{Q}_e \rangle$ and $\langle \mathbf{Q}_h, \mathbf{Q}_h \rangle$ denote the squared lengths of the charge vectors $\mathbf{Q}_e$ and $\mathbf{Q}_h$ which are integer-valued linear functionals on the Euclidean lattices Γ_e and Γ_h, respectively. We remark that, by assumption (A5), it suffices to focus our attention on the analysis of, say, the electron-rich subfluids of a QH fluid and the corresponding CQHL. In the following, we drop the subscript e from our notation.

Definition. A chiral quantum Hall lattice (CQHL) is a pair $(\Gamma, \mathbf{Q})$, where Γ is an odd, integral, Euclidean lattice and $\mathbf{Q}$ is an odd, primitive vector in Γ^*, the dual lattice of Γ.

We clarify this definition by recalling some technical notions:

1. Let V denote a real, N-dimensional vector space with inner product (or metric) $\langle\ ,\ \rangle$. We choose an *integral basis*, $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ in V.

Integrality means that the (regular, symmetric) matrix of scalar products $K=(K_{ij})$, the so-called associated *Gram matrix*, is integral, i.e.,

$$K_{ij} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle \in \mathbb{Z} \qquad \text{for all} \quad i, j = 1, \ldots, N \tag{2.2}$$

Taking integral linear combinations of these basis vectors, we can form the *integral lattice*

$$\Gamma := \left\{ \mathbf{q} \in V \,|\, \mathbf{q} = \sum_{i=1}^{N} q^i \mathbf{e}_i,\ q^i \in \mathbb{Z}, \text{ for all } i = 1, \ldots, N \right\} \tag{2.3}$$

A lattice Γ is said to be *Euclidean* if the metric $\langle\ ,\ \rangle$ is positive-definite (i.e., its Gram matrix K is a positive-definite matrix).

Introducing the *dual basis* $\{\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^2\}$ which is characterized by the property that $\langle \boldsymbol{\varepsilon}^j, \mathbf{e}_i \rangle = \delta_i^j$ for all $i, j = 1, \ldots, N$ [i.e., $\boldsymbol{\varepsilon}^j = \sum_{i=1}^N (K^{-1})^{ji} \mathbf{e}_i$, $j = 1, \ldots, N$] we have the *dual lattice* Γ^* of the lattice Γ given by

$$\begin{aligned} \Gamma^* &:= \{ \mathbf{n} \in V \,|\, \langle \mathbf{n}, \mathbf{q} \rangle \in \mathbb{Z} \text{ for all } \mathbf{q} \in \Gamma \} \\ &= \left\{ \mathbf{n} \in V \,\middle|\, \mathbf{n} = \sum_{j=1}^{N} n_j \boldsymbol{\varepsilon}^j,\ n_j \in \mathbb{Z}, \text{ for all } j = 1, \ldots, N \right\} \supseteq \Gamma \end{aligned} \tag{2.4}$$

2. We recall Kramer's rule,

$$(K^{-1})^{ij} = \frac{1}{\Delta} \tilde{K}^{ij} \tag{2.5}$$

where $\tilde{K}$ denotes the cofactor (or adjoint) matrix of K and $\Delta := \det K$ denotes the *discriminant* of the lattice Γ. We note that Δ is the order of the Abelian group Γ^*/Γ, or, from a geometrical point of view, it specifies the relative size of the lattice Γ when viewed as a sublattice of the dual lattice Γ^*.

3. An integral lattice Γ is said to be *odd* if it contains a vector $\mathbf{q}$ for which $\langle \mathbf{q}, \mathbf{q} \rangle$ is an odd integer. Thus Γ is odd if and only if K_{ii} is odd for at least one i in $1, \ldots, N$ (otherwise Γ is said to be even).

4. A vector $\mathbf{Q} = \sum_{j=1}^N Q_j \boldsymbol{\varepsilon}^j \in \Gamma^*$ is called *primitive* (or *visible*) if the greatest common divisor (gcd) of its dual components Q_j equals unity, i.e.,

$$\gcd(Q_1, \ldots, Q_N) = \gcd(\langle \mathbf{Q}, \mathbf{e}_1 \rangle, \ldots, \langle \mathbf{Q}, \mathbf{e}_N \rangle) = 1 \tag{2.6}$$

Geometrically, $\mathbf{Q} \in \Gamma^*$ is primitive means that the line segment from the origin to $\mathbf{Q}$ does not contain any point of Γ^* other than 0 and $\mathbf{Q}$.

5. The vector $\mathbf{Q} \in \Gamma^*$ is said to be *odd* if the following congruence holds

$$\langle \mathbf{Q}, \mathbf{q} \rangle \equiv \langle \mathbf{q}, \mathbf{q} \rangle \bmod 2 \qquad \text{for all} \quad \mathbf{q} \in \Gamma \tag{2.7}$$

6. A lattice Γ is called *decomposable* (or *composite*) if it can be written as an orthogonal direct sum of sublattices,

$$\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_k \qquad \text{for some} \quad k \geqslant 2 \tag{2.8}$$

i.e., for arbitrary vectors $\mathbf{q}_i \in \Gamma_i$ and $\mathbf{q}_j \in \Gamma_j$ we have $\langle \mathbf{q}_i, \mathbf{q}_j \rangle = 0$ for all $i \neq j$. Otherwise Γ is said to be *indecomposable.* If $(\Gamma, \mathbf{Q})$ is a composite CQHL with decomposition (2.8), then the dual lattice has the associated decomposition $\Gamma^* = \Gamma_1^* \oplus \Gamma_2^* \oplus \cdots \oplus \Gamma_k^*$ and the corresponding decomposition of the charge vector reads $\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2 + \cdot + \mathbf{Q}_k$. The decomposition (2.8) is reflected in the formula

$$\sigma_{\mathrm{H}} = \langle \mathbf{Q}, \mathbf{Q} \rangle = \langle \mathbf{Q}_1, \mathbf{Q}_2 \rangle + \cdots + \langle \mathbf{Q}_k, \mathbf{Q}_k \rangle = \sigma_{\mathrm{H}}^1 + \cdots + \sigma_{\mathrm{H}}^k \tag{2.9}$$

From a physical point of view, *indecomposable* CQHLs can be considered as describing "elementary" QH fluids, and for this reason we mainly focus on indecomposable lattices in the present work. We note that, as suggested by (2.1), we can think of QH fluids with electron- *and* hole-rich subfluids as being described by particular composite lattices, namely ones that are orthogonal direct sums of two CQHLs of *opposite* chirality (i.e., there are currents of both chiralities circulating at the edge of these fluids).

7. We introduce two physically natural restrictions on chiral QH lattices. First, let $(\Gamma, \mathbf{Q})$ be a decomposable CQHL with decomposition (2.8) and (2 9). Then $(\Gamma, \mathbf{Q})$ is said to be *proper* if no component $\mathbf{Q}_j$, $j = 1, \ldots, k$, of the charge vector $\mathbf{Q}$ is vanishing. Note that if, say, $\mathbf{Q}_j = 0$, then $\sigma_{\mathrm{H}}^j = 0$ in (2.9), and the subfluid corresponding to $(\Gamma_j, \mathbf{Q}_j)$ does not have any interesting electric properties [see also the remark after (1.12)]. For this reason we neglect improper CQHLs in the present work, and properness will always be understood to hold in the sequel.

Second, from a physical point of view it is natural to even sharpen the notion of properness as follows: Let $(\Gamma, \mathbf{Q})$ be a decomposable CQHL as above. Then $(\Gamma, \mathbf{Q})$ is said to be primitive if every component $\mathbf{Q}_j$, $j = 1, \ldots, k$, of the charge vector $\mathbf{Q}$ is a primitive vector in Γ_j^*; see (2.6).

Primitive CQHLs are proper, but the contrary is not necessarily true. We will show in Theorem 4.6 in Section 4 that, for a subclass of proper CQHLs, the contrary can be inferred. Moreover, note that indecomposable

CQHLs are proper and primitive. *The classification of primitive CQHLs is the main objective of the present paper, and the corresponding results are given in Sections* 4–6.

We remark that, as explained in Appendix E, all *chiral* hierarchy fluids correspond to primitive CQHLs. In general, however, there are (nonchiral) hierarchy fluids which are associated with *nonprimitive* CQHLs. For some examples, see (b) in Appendix E. We do not find these nonprimitive proposals very attractive and shall provide, at some of the corresponding Hall fractions, primitive CQHLs in Sections 5 and 6.

QH Lattice–QH Fluid Dictionary. We briefly summarize the basic relationship between the language of QH lattices and the description of physical properties of the corresponding QH fluids; see Section 1, and, for a detailed discussion, refs. 5–7.

Let $(\Gamma, \mathbf{Q})$ denote a CQHL. Then any vector $\mathbf{q}$ in the lattice Γ labels a *multielectron* or *multihole excitation* above the ground state of the corresponding QH fluid which is localized in some bounded region of the plane of the system. (Here "hole" means a "missing electron" in an electron-rich fluid.) Next, *arbitrary* localized physical excitations of the QH fluid (*quasiparticles*) are labeled by vectors $\mathbf{n}$ that form a lattice Γ_{phys} which clearly has to contain Γ and which itself is contained in or is equal to Γ^*:

$$\Gamma \subseteq \Gamma_{\text{phys}} \subseteq \Gamma^* \tag{2.10}$$

In our units where $e = -1$ the *total electric charge* $q_{\text{el}}(\mathbf{n})$ of a physical excitation labeled by $\mathbf{n} \in \Gamma_{\text{phys}}$ is given by the inner product of $\mathbf{n}$ with the charge vector $\mathbf{Q}$,

$$q_{\text{el}}(\mathbf{n}) = \langle \mathbf{Q}, \mathbf{n} \rangle \tag{2.11}$$

and the *statistical phase* $\vartheta(\mathbf{n})$ of the excitation is determined by the squared length (modulo 2) of $\mathbf{n}$,

$$\vartheta(\mathbf{n}) \equiv \langle \mathbf{n}, \mathbf{n} \rangle \bmod 2 \tag{2.12}$$

We note that (2.12) corresponds to a normalization of the statistical phase such that bosons have $\vartheta \equiv 0 \pmod 2$ while fermions have $\vartheta \equiv 1 \pmod 2$. As mentioned in Section 1, moving (adiabatically) one quasiparticle labeled by a vector $\mathbf{n}_1$ around another one labeled by a vector $\mathbf{n}_2$ along a counterclockwise oriented loop, the state vector describing the system changes by a phase factor $\exp(2\pi i \langle \mathbf{n}_1, \mathbf{n}_2 \rangle)$; see (1.17).

Examples. We conclude this section by describing the two most basic examples of QH fluids in the language of QH lattices introduced above:

(a) *QH fluids with* $\sigma_H = N$, $N = 1, 2, \ldots$, *in the noninteracting electron approximation.* These integer QH fluids correspond to the (self-dual) unit Euclidean lattices in N dimensions: $\Gamma = \Gamma_{\text{phys}} = \Gamma^* = \mathbb{Z}^N = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. Here N is the number of separately conserved edge currents[8] or filled Landau levels. Denoting by $\mathbf{e}_i$ the generator of the ith summand, $i = 1, \ldots, N$, we have $K_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. By the primitivity condition on the charge vector $\mathbf{Q}$ (see point 7 above), we have $\mathbf{Q} = \mathbf{e}_1 + \cdots + \mathbf{e}_N$ and $\langle \mathbf{Q}, \mathbf{Q} \rangle = 1 + \cdots + 1 = N$. Finally we note that, by the self-duality of $\mathbb{Z}^N$ there are no fractionally charged excitations with fractional statistics ("anyons") in these fluids.

(b) *The Laughlin fluids*[35] at $\sigma_H = 1/m$, *where* $m = 2p + 1$, $p = (0)$, $1, 2, \ldots$. Here, $\Gamma = \sqrt{m}\,\mathbb{Z}$, which is the one-dimensional lattice generated by $\mathbf{e}$ with squared length $\langle \mathbf{e}, \mathbf{e} \rangle = m$. The dual lattice is $\Gamma^* = (1/\sqrt{m})\,\mathbb{Z}$, which is generated by $\boldsymbol{\varepsilon} = \mathbf{e}/m$. The charge vector, being primitive in Γ^*, takes the form $\mathbf{Q} = \boldsymbol{\varepsilon}$, and thus $\sigma_H = \langle \mathbf{Q}, \mathbf{Q} \rangle = 1/m$. The quasiparticles are labeled by $\xi\boldsymbol{\varepsilon} \in \Gamma_{\text{phys}} = \Gamma^*$, $\xi \in \mathbb{Z}$. By (2.11), they have fractional electric charges $q_{\text{el}}(\xi) = \langle \mathbf{Q}, \xi\boldsymbol{\varepsilon} \rangle = \xi/m$, and by the congruence (2.12), they have fractional statistical phases $\vartheta(\xi) \equiv \langle \xi\boldsymbol{\varepsilon}, \xi\boldsymbol{\varepsilon} \rangle \equiv \xi^2/m \pmod 2$.

Note that, in this case, the knowledge of the electric charges q_{el} of the excitations uniquely determines their statistical phases ϑ. Such a charge–statistics relation is a property of many interesting higher dimensional QH lattices; see Theorem 4.5. However, such a relation does *not* hold for arbitrary QH lattices.

3. BASIC INVARIANTS OF CHIRAL QH LATTICES (CQHLS) AND THEIR PHYSICAL INTERPRETATIONS

Invariants of CQHLs, most of which appear to be new, provide physically interesting information about the corresponding chiral (i.e., electron- *or* hole-rich) QH (sub)fluids. Most of the invariants summarized below have been introduced in ref. 6, where more details can be found. From the classification results presented in Sections 5 and 6 and from the discussion in Section 7, it follows that a microscopic understanding and a corresponding determination of the values of these invariants pose interesting open problems in the theory of the QH effect.

The invariants of a (proper) CQHL $(\Gamma, \mathbf{Q})$ capture *intrinsic* properties of $(\Gamma, \mathbf{Q})$, i.e., properties that do not depend on the explicit choice of a basis in Γ and on the "reshuffling" of electric charge assignments in the lattice corresponding to a transformation of $\mathbf{Q}$ by an orthogonal symmetry of Γ. Choosing a basis, $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ in Γ, we specify the CQHL by the (integral) Gram matrix $K_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$, $i, j = 1, \ldots, N$, and by the row vector

$\underset{\rightarrow}{Q} = (Q_1, \ldots, Q_N)$ which specifies the components of the charge vector $\mathbf{Q}$ in the associated dual basis $\{\varepsilon^1, \ldots, \varepsilon^N\}$ of Γ^*, i.e., $\mathbf{Q} = \sum_{j=1}^{N} Q_j \varepsilon^j$; see Eqs. (2.2)–(2.4). Choosing a different basis in Γ, we have the resulting pair $(K', \underset{\rightarrow}{Q}')$ related to the pair $(K, \underset{\rightarrow}{Q})$ by

$$K' = S^T K S \qquad \text{and} \qquad \underset{\rightarrow}{Q}' = \underset{\rightarrow}{Q} S \tag{3.1}$$

where S is an element in $GL(N, \mathbb{Z})$, the group of integral, nondegenerate $N \times N$-matrices. Note that for S^{-1} to be an element of the group, the determinant of any element S has to be ± 1.

Following the proposal in ref. 6 a concise presentation of the numerical invariants of a CQHL $(\Gamma, \mathbf{Q})$ is given by the associated *symbol*

$${}_N\!\left(\sigma_H = \frac{n_H}{d_H}\right)_\lambda^g [l_{\min}, l_{\max}] \tag{3.2}$$

where the invariants summarized in the symbol have the following mathematical definitions and physical interpretations:

1. $N := \dim \Gamma = \operatorname{rank} \Gamma$; the *lattice dimension* N gives (in the scaling limit) the number of separately conserved $u(1)$-current densities in the corresponding QH fluid. Although no rigorous results are known, we expect N to depend on the filling factor and on the density or strength of impurities (disorder) in the system. We expect that the upper bound N_* on the dimension N of physically realizable CQHLs tends to ∞ as the density or strength of impurities tends to 0.[7]

2. By (2.1) and (3.1) the *Hall conductivity* (or *Hall fraction*) σ_H is clearly a CQHL invariant: $\sigma_H = \langle \mathbf{Q}, \mathbf{Q} \rangle = \underset{\rightarrow}{Q} \cdot K^{-1} \underset{\rightarrow}{Q}^T$. By (2.5) and the definition of $\underset{\rightarrow}{Q}$ it is a positive *rational* number.

3. Writing $\sigma_H = n_H / d_H$ with $\gcd(n_H, d_H) = 1$, we can write the important invariant of the lattice given by its *discriminant* Δ as

$$\Delta := \det K = l d_H \tag{3.3}$$

where the invariant l is called the *level* of the CQHL $(\Gamma, \mathbf{Q})$; see (2.5).

4. The level l satisfies an interesting factorization property, namely $l = g\lambda$, where g is defined by $g := \gcd(Q^1, \ldots, Q^N)$ with $Q^j := \Delta \langle \mathbf{Q}, \varepsilon^j \rangle$, $j = 1, \ldots, N$ and $\{\varepsilon^1, \ldots, \varepsilon^N\}$ *any* dual basis of Γ^*. Thus, by (3.3), the discriminant is given by $\Delta = g\lambda d_H$. The invariant λ is called the *charge parameter* and its physical relevance derives from the following fact: one can prove[6] that, in our units where $e = -1$, the smallest possible (fractional) electric

charge of a quasi-particle excited above the ground state in the corresponding QH fluid is given by

$$e^* := \min_{\mathbf{n} \in \Gamma^*, \langle \mathbf{Q}, \mathbf{n} \rangle \neq 0} |\langle \mathbf{Q}, \mathbf{n} \rangle| = \frac{1}{\lambda d_{\mathrm{H}}} \tag{3.4}$$

5. Finally we provide definitions of the *relative-angular-momentum invariants* $l_{\min}$ and $l_{\max}$. Since $\mathbf{Q}$ is a primitive vector in Γ^* [see (2.6)], there is a basis of Γ, $\{\mathbf{q}_1,..., \mathbf{q}_N\}$, such that $\langle \mathbf{Q}, \mathbf{q}_i \rangle = 1$, $i = 1,..., N$. The set of all such "symmetric" bases is denoted by $\mathscr{B}_{\mathbf{Q}}$. Then, for any CQHL $(\Gamma, \mathbf{Q})$ we define the invariants

$$L_{\min} := \min_{\mathbf{q} \in \Gamma, \langle \mathbf{Q}, \mathbf{q} \rangle = 1} \langle \mathbf{q}, \mathbf{q} \rangle \tag{3.5}$$

and

$$L_{\max} := \min_{\{\mathbf{q}_1,..., \mathbf{q}_N\} \in \mathscr{B}_{\mathbf{Q}}} \left(\max_{1 \leqslant i \leqslant N} \langle \mathbf{q}_i, \mathbf{q}_i \rangle \right) \tag{3.6}$$

In the situation where $(\Gamma, \mathbf{Q})$ is a *primitive decomposable* CQHL with decomposition $(\Gamma, \mathbf{Q}) = \bigoplus_{j=1}^{k} (\Gamma_j, \mathbf{Q}_j)$ [see (2.8)] it is natural to refine the definitions (3.5) and (3.6) as follows:

$$l_{\min}(\Gamma, \mathbf{Q}) := \min_{1 \leqslant j \leqslant k} L_{\min}(\Gamma_j, \mathbf{Q}_j) \geqslant L_{\min}(\Gamma, \mathbf{Q}) \tag{3.7}$$

and

$$l_{\max}(\Gamma, \mathbf{Q}) := \max_{1 \leqslant j \leqslant k} L_{\max}(\Gamma_j, \mathbf{Q}_j) \geqslant L_{\max}(\Gamma, \mathbf{Q}) \tag{3.8}$$

We note that, by the oddness of $\mathbf{Q}$ [see (2.7)], the relative-angular-momentum invariants (3.5)–(3.8) are positive, *odd* integers which satisfy

$$L_{\min} \leqslant L_{\max}, \qquad l_{\min} \leqslant l_{\max} \tag{3.9}$$

Exploiting the Chern–Simons description of QH fluids, it has been argued in refs. 6 and 7 that, physically, for an *elementary* chiral QH fluid corresponding to the *indecomposable* CQHL $(\Gamma, \mathbf{Q})$, $l_{\min} = L_{\min}$ indicates the smallest possible relative angular momentum of two electrons excited above the ground state of the fluid. The physical relevance of the quantity $l_{\max}$ as well as its role in the classification of CQHLs will be expounded in great detail in Sections 4–6.

If the values of the quantities $l_{\min}$ and $l_{\max}$ are clear from context, they will be droped from the symbol (3.2).

Note that the elementary invariants in points 1–4 are clearly well defined also for (general) QH lattices.[6]

Examples. We illustrate the above invariants by considering some examples.

(a) The integer QH fluids discussed at the end of Section 2 (noninteracting electron systems) are characterized by the symbols

$$ {}_N\!\left(\frac{n_{\mathrm{H}}}{d_{\mathrm{H}}}\right)^{g}_{\lambda} [l_{\min}, l_{\max}] = {}_N(N)^{1}_{1}[1, 1], \qquad N = 1, 2,... \tag{3.10} $$

Note that, by the decomposability of the corresponding CQHLs, we can write ${}_N(N)^{1}_{1} = {}_1(1)^{1}_{1} \oplus \cdots \oplus {}_1(1)^{1}_{1}$ in accordance with the physical picture of N independent, filled Landau levels.

(b) The Laughlin fluids, also discussed at the end of Section 2, correspond to CQHLs for which the associated symbols read

$$ {}_N\!\left(\frac{n_{\mathrm{H}}}{d_{\mathrm{H}}}\right)^{g}_{\lambda} [l_{\min}, l_{\max}] = {}_1\!\left(\frac{1}{2p+1}\right)^{1}_{1} [2p+1, 2p+1], \qquad p = 1, 2,... \tag{3.11} $$

For a discussion of the special status of the Laughlin fluids from a classification point of view, see Theorems 4.4 and 4.8 in Section 4.

(c) For each $p = 1, 2,...,$ there is the series of Hall fractions $\sigma_{\mathrm{H}} = N/(2pN+1)$ with $N = 1, 2,...$. From the data presented in Fig. 1, it is clear that many of the experimentally most prominent Hall fractions belong to these series (or to the charge-conjugated partner series of the one with $p = 1$; see the discussion in Section 7). We note that these fractions also figure prominently in Jain's work[31]—the basis of the Jain–Goldman hierarchy scheme[30]—and we refer to Theorem 4.8 in Section 4, where, from a classification point of view, the uniqueness of the associated CQHLs is discussed. The above series of Hall fractions can be obtained by the following series of indecomposable CQHLs: the data pairs $(K, \underset{\rightarrow}{Q})$ which determine these CQHLs are given, in some bases that we call "normal," by

$$ K = \left.\left(\begin{array}{c|ccccc} 2p+1 & -1 & 0 & \cdot & \cdot & 0 \\ \hline -1 & 2 & -1 & 0 & \cdot & 0 \\ 0 & - & 2 & -1 & \cdot & \cdot \\ \cdot & 0 & -1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ 0 & 0 & \cdot & 0 & - & 2 \end{array}\right)\right\} N, \qquad \underset{\rightarrow}{Q} = (1, \underbrace{0,..., 0}_{N}) \tag{3.12} $$

and the associated symbols read

$$
{}_N\left(\frac{n_{\mathrm{H}}}{d_{\mathrm{H}}}\right)_{\lambda}^{g} [l_{\min}, l_{\max}] = {}_N\left(\frac{N}{2pN+1}\right)_{1}^{1} [2p+1, 2p+1], \qquad p = 1, 2,... \tag{3.13}
$$

Note that the $(N-1)$-dimensional submatrix in the lower right of K is the Cartan matrix of the simple Lie algebra $A_{N-1} = su(N)$, $N = 2, 3,...$; see Appendix A. For $N = 1$, we recognize in (3.12) and (3.13) the expressions corresponding to the Laughlin fluids; see example (b). In connection with the QH effect the matrices in (3.12) first appeared in ref. 9. Combining results of refs. 9 and 3 (see Appendix E), we see that the CQHLs specified by (3.12) correspond to the "basic Jain states"[31] at $\sigma_{\mathrm{H}} = N/(2pN+1)$. Moreover, it has been shown in ref. 3 that the QH fluids corresponding to (3.12) exhibit large symmetries, namely $su(N)$-current algebras at level 1; see also ref. 5. In Section 5 we show that the above CQHLs belong to an interesting class of CQHLs with "large" symmetries, the so-called "maximally symmetric" CQHLs. The classification of "maximally symmetric" CQHLs will be the main objective of Section 5.

We note that, by extending definitions (3.12) and (3.13) to $p = 0$, the *composite* integer QH fluids of example (a) can be included as special cases of (c).

4. GENERAL THEOREMS AND CLASSIFICATION RESULTS FOR CQHLS

The purpose of this section is to review general facts and classification results for CQHLs in order to put the more specific classification results given in Sections 5 and 6 into a broader perspective. We summarize, in the form of eight theorems, results that have been presented in our previous work[6, 38, 7] where more details can be found. We indicate those proofs that have not been given previously. Moreover, we discuss phenomenological implications of our theorems.

The first two theorems are based on *CQHL inequalities* that establish useful relations between some of the numerical invariants introduced in Section 3.

Theorem 4.1. The set of (proper) CQHLs $(\Gamma, \mathbf{Q})$ with dimensions $N \leqslant N_*$ and relative-angular-momentum invariants $l_{\max} \leqslant l_*$, where N_* and l_* are two given integers, is *finite*.

This theorem implies that the set of Hall fractions σ_{H} that can be realized by CQHLs which satisfy the above bounds on N and $l_{\max}$ is finite.

We remark, however, that the number of possible fractions is growing superexponentially fast in N_* and l_*, e.g., for $N_* = 2$ and $l_* = 3$, there are ten CQHLs, while for $N_* = 3$ and $l_* = 5$, one finds already more than 250 CQHLs. Fortunately, in the physically relevant situation where one also has a natural upper bound σ_* on the Hall fractions to be considered, the number of CQHLs satisfying this bound and the ones in Theorem 4.1 is drastically reduced! This fact is illustrated by the classification results in Sections 5 and 6.

The basic tools in proving Theorem 4.1 are Hadamard's inequality for positive-definite quadratic forms (see, e.g., ref 39), which implies that

$$\lambda g\, d_{\mathrm{H}} = \Delta = \det K \leqslant l_{\max}^{N} \tag{4.1}$$

and the fact that[42]

$$\lambda g n_{\mathrm{H}} \leqslant C(N)\, l_{\max}^{n-1} \tag{4.2}$$

where $C(N)$ is a constant depending on the lattice dimension N, e.g., for two-dimensional CQHLs, one finds that $C(2) = 4$. ▮

Physically, N is the number of separately conserved $u(1)$-current densities in a QH fluid (in the scaling limit). A larger amount of disorder (an increased density or strength of impurities) in the system is expected to reduce the quantity N because of "channel-mixing" effects. Hence it is natural to impose an upper bound N_* *depending on disorder* on the dimension N of physically relevant CQHLs. With respect to an upper bound on the relative-angular-momentum invariant $l_{\max}$ we argue, physically, that if $l_{\max}$ were too large, then the density of electrons in the ground state of a (pure) system would be so small that it would be energetically more favorable for the electrons to form a Wigner crystal, thereby destroying the incompressibility of the system; see ref. 40 and, for a review of recent experiments, ref. 41. Given this remark the following basic CQHL inequality is of interest.

Theorem 4.2. For a CQHL $(\Gamma, \mathbf{Q})$ the Hall fraction σ_{H} and the relative-angular-momentum invariants $L_{\min}$, $l_{\min}$, and $l_{\max}$ satisfy

$$1/\sigma_{\mathrm{H}} \leqslant L_{\min} \leqslant l_{\min} \leqslant l_{\max} \tag{4.3}$$

This theorem is a direct consequence of the Cauchy–Schwarz inequality (in the real vector space $V \supset \Gamma$), $\langle \mathbf{Q}, \mathbf{q} \rangle^2 \leqslant \langle \mathbf{q}, \mathbf{q} \rangle \langle \mathbf{Q}, \mathbf{Q} \rangle$, and the fact that, for any vector $\mathbf{q} \in \Gamma$, with $q_{\mathrm{el}}(\mathbf{q}) = \langle \mathbf{Q}, \mathbf{q} \rangle \neq 0$ we have $\langle \mathbf{Q}, \mathbf{q} \rangle^2 \geqslant 1$. ▮

If we suppose that, physically, chiral QH fluids satisfy a universal bound $l_{\max} \leqslant l_*$ then (4.3) tells us that CQHLs with $\sigma_H < 1/l_*$ are physically *irrelevant*. Note that the data in Fig. 1 are consistent with a choice of $l_* = 7$.

Given these observations on the quantities N and $l_{\max}$ one is led to the following heuristic principle:

Stability Principle. The smaller the values of the CQHL invariants N and $l_{\max}$, the more stable the corresponding chiral QH fluid.

This heuristic stability principle will receive further support when comparing our classification results of Sections 5 and 6 with the experimental data of Fig. 1; see Fig. 2 and the discussion in Section 7, where an even sharper version is proposed.

Theorem 4.3. Let $(\Gamma, \mathbf{Q})$ be a CQHL with *even* Hall denominator d_H. Then the charge parameter λ has to be *even*, too.

For a proof of this theorem, we first define the vector $\mathbf{v} := \lambda d_H \mathbf{Q} \in \Gamma^*$. Then, for all dual vectors $\mathbf{n} = \sum_{j=1}^{N} n_j \boldsymbol{\varepsilon}^j \in \Gamma^*$ we find that

$$\langle \mathbf{v}, \mathbf{n} \rangle = \lambda d_H \langle \mathbf{Q}, \mathbf{n} \rangle = \sum_{j=1}^{N} (\Delta \langle \mathbf{Q}, \boldsymbol{\varepsilon}^j \rangle / g) \, n_j = \sum_{j=1}^{N} (Q^j/g) \, n_j \in \mathbb{Z}$$

by using $\Delta = g\lambda d_H$ and the definition of g; see points 3 and 4 in Section 3. Thus $\mathbf{v}$ is actually an element of $(\Gamma^*)^* \simeq \Gamma$ and hence, by the oddness of $\mathbf{Q}$ [see (2.7)], the congruence $\langle \mathbf{Q}, \mathbf{v} \rangle \equiv \langle \mathbf{v}, \mathbf{v} \rangle \pmod 2$ holds. Now, by the definitions of σ_H and $\mathbf{v}$ it follows that the l.h.s. of the congruence equals λn_H and the r.h.s. equals $\lambda^2 d_H n_H$, i.e., $\lambda n_H \equiv \lambda^2 d_H n_H \pmod 2$. Finally, since for d_H even, n_H is odd (see point 3 in Section 3), the latter congruence would be false if λ were odd. ▮

The phenomenologically interesting implication of Theorem 4.3 is that, in QH fluids with an even Hall denominator d_H, one predicts the existence of quasiparticle excitations above the ground state with "fractional" fractional charges, i.e., since $\lambda = 2, 4, \ldots$,

$$e^* = 1/(\lambda d_H) \leqslant 1/(2 d_H) \tag{4.4}$$

It would be interesting to test this model-independent prediction experimentally for even-denominator QH fluids at $\sigma_H = 1/2$ and $5/2$ mentioned in Section 1: we predict that $e^* \leqslant 1/4$ (in units where $e = -1$)!

Theorem 4.4. At every Hall fraction $\sigma_H = 1/m$, m odd, there is a *unique* indecomposable CQHL with the property that its level $l = \lambda g = 1$.

This CQHL is one-dimensional and corresponds to the Laughlin fluid at $\sigma_H = 1/m$. Moreover, any CQHL with $\sigma_H = 1/m$, m odd, and $N \geqslant 2$ has a charge parameter $\lambda \geqslant 2$.

A proof of this theorem is given in ref. 6, Section 7.5.

The CQHLs corresponding to the Laughlin fluids have been described explicitly in example (b) at the end of Section 2. We emphasize that, by an argument similar to the one in (4.4), the last statement in Theorem 4.4 has implications that are, in principle, observable! In Section 7, an example illustrating this point is discussed when analyzing possible phase transitions at $\sigma_H = 1$.

An interesting subclass of CQHLs is formed by CQHLs with level $l = 1$, i.e., their lattice discriminant Δ equals the Hall denominator d_H. Indecomposable CQHLs with level $l = 1$, and thus $\lambda = g = 1$, have been classified for $d_H \leqslant 25$ and N below relatively high "critical" dimensions $N_c(\sigma_H)$, typically around 10.[38, 6]

This subclassification has been achieved by combining the recent lattice-classification results of Conway and Sloane[43] with a systematic investigation of the possible charge vectors $\mathbf{Q}$ in the duals of all the classified (odd, integral, Euclidean) lattices. For the latter search one makes use of the following fact: from the Cauchy–Schwarz inequality and the defining relation $\sigma_H = \langle \mathbf{Q}, \mathbf{Q} \rangle$ one infers that for a CQHL $(\Gamma, \mathbf{Q})$ the dual components Q_j of the charge vector $\mathbf{Q} = \sum_{j=1}^{N} Q_j \boldsymbol{\varepsilon}^j$ are constrained by

$$Q_j^2 \leqslant \sigma_H l_{\max} \qquad \text{for all} \quad j = 1, \ldots, N \tag{4.5}$$

Thus, restricting one's focus to CQHLs with $l_{\max} \leqslant l_*$ and $\sigma_H \leqslant \sigma_*$ one finds that (4.5) implies that the search for all possible charge vectors $\mathbf{Q}$ in the dual of a given lattice Γ is a *finite* problem. ▌

In the next theorem we recall a few general properties of CQHLs with level $l = 1$; for proofs see ref. 6.

Theorem 4.5. Let $(\Gamma, \mathbf{Q})$ be a (proper) CQHL with level $l = \lambda g = 1$. Then: (i) d_H is *odd*, and $\Gamma^*/\Gamma \sim \mathbb{Z}_{d_H}$; (ii) in order to realize a Hall fraction σ_H with n_H even (odd), N has to be even [odd, and $N \equiv n_H \pmod 4$]; (iii) for quasiparticles labeled by $\mathbf{n} \in \Gamma^*$ a charge–statistics relation holds: if $q_{el}(\mathbf{n}) = \varepsilon/d_H$, then $\vartheta(\mathbf{n}) \equiv (n_H)^{-1} \varepsilon^2/d_H \pmod 2$.

In the last statement of this theorem, the number $(n_H)^{-1}$ is defined as follows: if n_H is odd, then $n_H(n_H)^{-1} \equiv 1 \pmod{2d_H}$, and if n_H is even, then $n_H := 2(2n_H)^{-1} + d_H$, with $2n_H(2n_H)^{-1} \equiv 1 \pmod{d_H}$. A proof of this theorem can be found in ref. 6, Section 5.

Shift Maps and their Implications. In the remaining part of this section we study "structurally similar" chiral QH fluids. At the level of CQHLs "structural" relationships are realized by particular maps called *shift maps*. From a classification point of view, shift maps allow us, under suitable conditions, to immediately carry over classification results for CQHLs with Hall fractions in a given interval to corresponding results for other intervals. Phenomenologically interesting implications of structural relationships are outlined at the end of this section and in Section 7.

First we divide the interval $(0, \infty)$ of possible Hall fractions σ_H into a sequence of suitable subintervals, "*windows*," $\Sigma_p^{\pm}$ defined by

$$\begin{aligned} \Sigma_p^+ &:= \{\sigma_H \mid 1/(2p+1) \leqslant \sigma_H < 1/(2p)\}, \qquad p = 1, 2,... \\ \Sigma_p^- &:= \{\sigma_H \mid 1/(2p) \leqslant \sigma_H < 1/(2p-1)\}, \qquad p = 1, 2,... \end{aligned} \tag{4.6}$$

The "+" superscripts in the window symbols Σ_p^+ are chosen because these subintervals contain the "first main series" of Hall fractions, $\sigma_H = N/(2pN+1)$, $N = 1, 2,...$. Similarly, the "−" superscripts for the "complementary" windows remind us that these windows contain the "second main series" of Hall fractions, $\sigma_H = N/(2pN-1)$, $N = 2, 3,...$. Moreover, we denote by Σ_0^+ the interval $[1, \infty)$ and by Σ_p the union of the two complementary subintervals Σ_p^+ and Σ_p^-, i.e., $\Sigma_p := \Sigma_p^+ \cup \Sigma_p^-$, $p = 1, 2,...$.

Second, we define a class of CQHLs that will figure prominently in the sequel.

Definition. A primitive CQHL $(\Gamma, \mathbf{Q})$ (see point 7 in Section 2) with Hall fraction $\sigma_H \in \Sigma_p$ is called *L-minimal* if $l_{\max}$ takes the smallest possible value consistent with (4.3), namely $l_{\max} = 2p+1$, $p = 1, 2,...$.

By (3.7)–(3.9), L-minimal CQHLs satisfy $L_{\min} = l_{\min} = L_{\max} = l_{\max} = 2p+1$. General, powerful implications that follow from L-minimality are summarized below in Theorems 4.6–4.8; for proofs, see ref. 7.

Theorem 4.6. For $p = 1, 2,...$, let $(\Gamma, \mathbf{Q})$ be a (proper) CQHL with $\sigma_H \in \Sigma_p$ and $L_{\max} = 2p+1$. Then $(\Gamma, \mathbf{Q})$ is primitive and L-minimal, i.e., we also have $L_{\min} = l_{\max} = 2p+1$. Moreover, $(\Gamma, \mathbf{Q})$ is *indecomposable* if $\sigma_H < 2/3$.

We note that the bound $\sigma_H < 2/3$ for indecomposability is sharp. As a matter of fact, at $\sigma_H = 2/3$ there is an L-minimal $(l_{\max} = 3)$ *composite* CQHL. It is given by the direct sum of two Laughlin fluids at $\sigma_H = 1/3$; see example (b) in Section 2.

Next, we give a precise definition of "shift maps."

Definition. Shift maps, denoted by $\mathscr{S}_p$, $p=1, 2,\ldots$, are maps between (proper) CQHLs of equal dimensions, $\mathscr{S}_p\colon (\Gamma, \mathbf{Q}) \mapsto (\Gamma', \mathbf{Q}')$. Starting from an arbitrary basis $\{\mathbf{e}_1,\ldots, \mathbf{e}_N\}$ of $(\Gamma, \mathbf{Q})$, the image $(\Gamma', \mathbf{Q}')$ is uniquely specified by constructing a basis $\{\mathbf{e}'_1,\ldots, \mathbf{e}'_N\}$ and a charge vector $\mathbf{Q}'$ that satisfy the conditions

$$\begin{aligned} K'_{ij} &= \langle \mathbf{e}'_i, \mathbf{e}'_j \rangle = \langle \mathbf{e}_i, \mathbf{e}_j \rangle + 2p \langle \mathbf{Q}, \mathbf{e}_i \rangle \langle \mathbf{Q}, \mathbf{e}_j \rangle \\ &= K_{ij} + 2p q_{\mathrm{el}}(\mathbf{e}_i)\, q_{\mathrm{el}}(\mathbf{e}_j) \\ Q'_i &= \langle \mathbf{Q}', \mathbf{e}'_i \rangle = \langle \mathbf{Q}, \mathbf{e}_i \rangle = Q_i \qquad \text{for all} \quad i, j = 1,\ldots, N \end{aligned} \tag{4.7}$$

Note that, although the conditions in (4.7) are formulated w.r.t. given bases, they specify the image $(\Gamma', \mathbf{Q}')$ uniquely, since different choices of bases and charge vectors in (4.7) simply lead to data pairs (K', Q') for $(\Gamma', \mathbf{Q}')$ which are all related by the equivalence transformations (3.1).

Denoting by $\Gamma_0 \subset \Gamma$ the *neutral sublattice* of a CQHL $(\Gamma, \mathbf{Q})$, i.e.,

$$\Gamma_0 := \{\mathbf{q} \in \Gamma \mid \langle \mathbf{Q}, \mathbf{q} \rangle = q_{\mathrm{el}}(\mathbf{q}) = 0\} \tag{4.8}$$

it is straightforward to show that shift maps leave neutral sublattices invariant,

$$\Gamma'_0 = \Gamma_0 \tag{4.9}$$

As will be explained in more detail in Section 5, Eq. (4.9) implies that (in the scaling limit) the corresponding chiral QH fluids exhibit the *same symmetries*. This equation is the mathematical basis for calling two chiral QH fluids *structurally similar*.

What is the action of the shift map $\mathscr{S}_p := (\Gamma, \mathbf{Q}) \mapsto (\Gamma', \mathbf{Q}')$, for $p = 1, 2,\ldots$, on the space of invariants introduced in Section 3?

(i) The discriminant Δ' of the (odd, integral, Euclidean) lattice Γ' is given by

$$\Delta' = \Delta(1 + 2p\sigma_{\mathrm{H}}) \tag{4.10}$$

(ii) The Hall conductivity changes according to

$$\frac{1}{\sigma'_{\mathrm{H}}} = \frac{1}{\sigma_{\mathrm{H}}} + 2p \tag{4.11}$$

which corresponds to the "D-operation" in the Jain–Goldman hierarchy scheme;[30] see also refs. 31 and 3. Note that Eq. (4.11) implies that any CQHL which is the image under a shift map $\mathscr{S}_p$, $p = 1, 2,\ldots$, necessarily has a Hall fraction strictly below $1/(2p)$.

(iii) The level l, g, and the charge parameter λ are all invariant under the action of a shift map $\mathscr{S}_p$.

We summarize (i)–(iii) by giving a succinct representation of the action of the shift map $\mathscr{S}_p$ at the level of the CQHL symbol,

$$ {}_{N}\left(\sigma_{\mathrm{H}}=\frac{n_{\mathrm{H}}}{d_{\mathrm{H}}}\right)_{\lambda}^{g} \xmapsto{\mathscr{S}_p} {}_{N}\left(\sigma'_{\mathrm{H}}=\frac{n_{\mathrm{H}}}{d_{\mathrm{H}}+2pn_{\mathrm{H}}}\right)_{\lambda}^{g}, \qquad p=1,2,... \tag{4.12} $$

(iv) The name "shift map" for $\mathscr{S}_p$ is motivated by the fact that the relative-angular-momentum invariants $L_{\min}$ and $L_{\max}$ are simply shifted by $2p$,

$$ L'_{\min}=L_{\min}+2p, \qquad L'_{\max}=L_{\max}+2p \tag{4.13} $$

Unfortunately, for the physically relevant invariants $l_{\min}$ and $l_{\max}$ of *generic* primitive CQHLs there does *not in general* hold a transformation rule similarly simple to (4.13)! Note, however, that for *indecomposable* CQHLs the identities $l_{\min}=L_{\min}$ and $l_{\max}=L_{\max}$ hold.

From the definitions above one sees that the shift maps $\mathscr{S}_p$ are invertible on the set of (proper) CQHLs with Hall fractions $\sigma_{\mathrm{H}}<1/(2p)$, $p=1,2,...$.—From (4.7) it follows that $\mathscr{S}_p^{-1}=\mathscr{S}_{-p}$.—The preimages of these CQHLs are readily seen to be (proper) CQHLs. The set of (proper) CQHLs is closed under the action of the maps $\mathscr{S}_p$ and their inverses.

However, the maps $\mathscr{S}_p$ and their inverses do *not* necessarily preserve the decomposability properties of CQHLs. (For example, composite CQHLs can be mapped into indecomposable ones, as illustrated in Theorem 4.8 below). Moreover, the maps $\mathscr{S}_p$ and their inverses do *not* in general preserve the primitivity property we have imposed on physically relevant composite CQHLs; see point 7 in Section 2. (For an example of a primitive CQHL with a preimage that is nonprimitive, see Section 4 in ref. 7.) From these remarks and the definitions (3.7) and (3.8) of the invariants $l_{\min}$ and $l_{\max}$, it is clear that the transformation properties of these invariants under shift maps are not as straightforward as the ones in (4.13).

We recall that the main objective of the present work is the classification of primitive CQHLs. Although this set is not closed under the action of shift maps and their inverses, it is remarkable that a subset of the primitive CQHLs, the class of *L-minimal* CQHLs [defined after (4.6)] is *closed* under the action of shift maps and their inverses. This is the key to powerful classification results that we state presently.

It is convenient to partition the class of L-minimal CQHLs into the following subsets:

$$\mathscr{H}_p^{\pm} := \{(\Gamma, \mathbf{Q}) \mid \sigma_{\mathrm{H}} \in \Sigma_p^{\pm}, L\text{-minimal}, \\ \text{i.e., primitive and } l_{\min} = l_{\max} = 2p+1\} \tag{4.14}$$

where $p = (0), 1, 2,...,$ in accordance with the definition of the windows $\Sigma_p^{\pm}$ given in (4.6).

The next two theorems show that, on the one hand, the sets $\mathscr{H}_p := \mathscr{H}_p^+ \cup \mathscr{H}_p^-$ are structurally similar for different p's, while, on the other hand, there is an essential structural asymmetry between the sets $\mathscr{H}_p^+$ and $\mathscr{H}_p^-$ for a given p.

Theorem 4.7. The sets $\mathscr{H}_p$ of L-minimal CQHLs with $\sigma_{\mathrm{H}} \in \Sigma_p$ for $p = 2, 3,...,$ are in *one-to-one correspondence* with the set $\mathscr{H}_1$. The corresponding bijections are realized by the shift maps $\mathscr{S}_{p-1}: \mathscr{H}_1 \to \mathscr{H}_p$.

The proof of this theorem rests on Theorem 4.6, and it should be emphasized that chirality and L-minimality are crucial for the theorem to hold[7]. Theorem 4.7 implies that, for the classification of L-minimal CQHLs, we can restrict our analysis to the lattices with Hall fractions σ_{H} in the "fundamental domain" $\Sigma_1 = [1/3, 1)$!

In ref. 7 the set $\mathscr{H}_0^+$ of L-minimal CQHLs with $\sigma_{\mathrm{H}} \in [1, \infty)$ has been constructed. Applying the shift map $\mathscr{S}_1$ to it, we obtain the set $\mathscr{H}_1^+$ of L-minimal CQHLs in the window $\Sigma_1^+ = [1/3, 1/2) \subset \Sigma_1$. Hence, by Theorem 4.7, all the sets $\mathscr{H}_p^+$, $p \geqslant 1$, are known. In fact, we have the following result.

Theorem 4.8. For each $p = 0, 1, 2,...,$ the set $\mathscr{H}_p^+$ of L-minimal CQHLs with $\sigma_{\mathrm{H}} \in \Sigma_p^+$ is uniquely given by the (infinite) series $N = 1, 2,...$ of maximally symmetric CQHLs with $SU(N)$-symmetry of N-ality 1, meaning that the one-electron states described by these CQHLs transform under the fundamental representation of $SU(N)$. For a given p the corresponding symbols read

$$_N\!\left(\sigma_{\mathrm{H}} = \frac{N}{2pN+1}\right)_1^1 [l_{\min} = l_{\max} = 2p+1], \qquad N = 1, 2,... \tag{4.15}$$

The maximally symmetric CQHLs of this theorem are N-dimensional and have been described explicitly in example (c) at the end of Section 3. In the notation of the next section [see (5.4)] the sets $\mathscr{H}_p^+$ are written as

$$\mathscr{H}_p^+ = \{(2p+1 \mid {}^1 A_{N-1}) \mid N = 1, 2,...\} \tag{4.16}$$

In Theorem 4.6 it has been stated that all CQHLs in (4.16) with $p > 0$ are *indecomposable*. Furthermore, since their level l equals unity, Theorem 4.5 states that a *charge–statistics relation* holds for the quasiparticle excitations of the corresponding QH fluids.

We conclude this section by discussing Table I, which summarizes the Hall fractions σ_H (with $d_H \leqslant 21$) in the windows Σ_p^+ that can or cannot be realized by elements in $\mathscr{H}_p^+$ with $p = 0$, 1, 2, and 3.

A first inspection of Table I reveals an impressive agreement between the Hall fractions predicted by L-minimal CQHLs and the experimentally observed values in the windows Σ_p^+, $p = 1, 2$, and 3. Note that CQHLs with higer dimensions and/or higher values of l_{max} are associated with less stable QH fluids, which is in accordance with our stability principle advocated at the beginning of this section. In the windows Σ_p^+, $p = 1, 2$, and 3, there is only one Hall fraction, 4/11, for which there are some experimental indications (however, only very weak ones!) that cannot be realized by an L-minimal CQHL.

Table I. Hall Fractions $\sigma_H \in \Sigma_p^+$ ($p = 0, 1, 2, 3$) That Are Uniquely Realizable or That Cannot Be Realized by an L-minimal CQHL[a]

$\Sigma_0^+ = [1, \infty)$, $l_{min} = l_{max} = 1$:	
Realizable:	$\bullet 1$ $\bullet 2$ $\bullet 3$ $\bullet 4$ $\bullet 5$ $\bullet 6$ $\bullet 7$ $\bullet 8$ $\bullet 9$ $\bullet 10$ ...
Not realizable:	All proper fractions
$\Sigma_1^+ = [1/3, 1/2)$, $l_{min} = l_{max} = 3$:	
Realizable:	$\bullet\frac{1}{3}$ $\bullet\frac{2}{5}$ $\bullet\frac{3}{7}$ $\bullet\frac{4}{9}$ $\bullet\frac{5}{11}$ $\bullet\frac{6}{13}$ $\cdot\frac{7}{15}$ $\circ\frac{8}{17}$ $\cdot\frac{9}{19}$ $\frac{10}{21}$
Not realizable:	$\frac{6}{17}$ $\cdot\frac{4}{11}$ $\frac{7}{19}$ $\frac{8}{21}$ $\frac{5}{13}$ $\frac{7}{17}$ $\frac{8}{19}$ and all even-denominator fractions
$\Sigma_2^+ = [1/5, 1/4)$, $l_{min} = l_{max} = 5$:	
Realizable:	$\bullet\frac{1}{5}$ $\bullet\frac{2}{9}$ $\circ\frac{3}{13}$ $\frac{4}{17}$ $\frac{5}{21}$
Not realizable:	$\frac{4}{19}$ and all even-denominator fractions
$\Sigma_3^+ = [1/7, 1/6)$, $l_{min} = l_{min} = 7$:	
Realizable:	$\circ\frac{1}{7}$ $\frac{2}{13}$ $\frac{3}{19}$
Not realizable:	All even-denominator fractions

[a] The symbols $\bullet$, $\circ$, and $\cdot$ specify the experimental status of the fractions as explained in Fig. 1. Fractions with $d_H > 21$ are omitted.

Concerning the results for the window Σ_0^+, we make three remarks. First, it is satisfying to see that the "standard" integer QH fluids of the non interacting electron approximation [see example (a) at the end of Section 2] are naturally included in our scheme and that they have a *unique* status. They correspond to the *L-minimal* CQHLs in the window Σ_0^+. We note that, contrary to the other CQHLs appearing in Table I, these integer CQHLs are *composite*.

Second, the result that in Σ_0^+ no proper Hall fraction can be realized by an L-minimal ($l_{\max} = 1$) leaves essentially only two ways open for realizing (in the scaling limit) a fractional QH fluid with $\sigma_{\mathrm{H}} > 1$: (i) as a composite system of independent, L-minimal electon- and/or hole-rich subfluids with partial Hall fractions $\sigma'_{\mathrm{H}} < 1$; see (2.1) and (2.9); physically, e.g., the natural idea of adding fully filled Landau levels to a fractional fluid with $\sigma'_{\mathrm{H}} < 1$ belongs to this situation; or (ii) as an indecomposable system described by a *non-L*-minimal CQHL (or a direct sum of such ones); see Section 6.

Third, since the inverse shift maps $\mathscr{S}_p^{-1}$ relate the CQHLs in the windows Σ_p^+, $p \geqslant 1$, to the ones in Σ_0^+, the results in Table I are reminiscent of Jain's construction[31] where interacting electron systems with $\sigma_{\mathrm{H}} \in \Sigma_p^+$ are related to noninteracting electron systems at the integers $N = \sigma_{\mathrm{H}}/(1 - 2p\sigma_{\mathrm{H}})$.

Given the discussion above, two questions emerge. First, what can we say about the CQHL class $\mathscr{H}_1^-$, and thus, by Theorem 4.7, about all sets $\mathscr{H}_p^-$ with $p \geqslant 1$? Second, given some experimental evidence for the Hall fraction 4/11, which cannot be realized by an L-minimal CQHL, we wish to get a fuller perspective on the assumption of L-minimality. Hence the question: how can we go beyond the classification of L-minimal CQHLs?

It turns out that already the first question, not to mention the second one, addresses a truly formidable task of great complexity! Section 5 provides a partial answer to the first question by classifying all "maximally symmetric," L-minimal CQHLs, which represent the most natural generalizations of the CQHLs appearing in Theorem 4.8. For low dimensions ($N \leqslant 4$), Section 6 gives the complete answer to the first question and makes the first manageable step in the direction of answering the second question.

5. CLASSIFICATION OF MAXIMALLY SYMMETRIC CQHLS

Maximally symmetric CQHLs correspond to the most natural generalizations of the "elementary" A- [or $su(N)$-] fluids that appeared in Theorem 4.8 of the last section and which have been shown to encompass the Laughlin fluids as well as the "basic" Jain fluids. Before we can give a precise definition of the class of maximally symmetric CQHLs, we need to investigate a general geometrical feature of CQHLs, namely their "Witt sublattices." We use technical language, and then translate our definitions into explicit statements at the level of the data pairs $(K, \vec{Q})$ associated with CQHLs; for the definition of these pairs, see the beginning of Section 3.

Let $(\Gamma, \mathbf{Q})$ be a CQHL. Then the *Witt sublattice* $\Gamma_{\mathrm{W}} \subset \Gamma$ is defined to be the sublattice of Γ generated by all vectors of length squared 1 and 2. The general theory of integral Euclidean lattices[43, 44] tells us that Γ_{W} is of the form

$$\Gamma_{\mathrm{W}} = \Gamma_A \oplus \Gamma_D \oplus \Gamma_l \oplus I_l \tag{5.1}$$

where I_l denotes the (self-dual) unit Euclidean lattice in l dimensions, and Γ_A, Γ_D, and Γ_E are direct sums of root lattices of the simple Lie algebras $A_{m-1} = su(m)$, $D_{m+2} = so(2m+4)$, $m = 2, 3,...,$ and E_m, $m = 6, 7, 8$, respectively. The subscripts in the symbols A_n, D_n, and E_n indicate the *ranks* of these algebras. We note that all the root lattices of these Lie algebras are generated by vectors of only one length, namely of length squared 2. (In the mathematical literature, the A-, D-, and E-Lie algebras are called simply-laced.)

Denoting by $\mathcal{O}$ the orthogonal complement of Γ_{W} in Γ, whose dimension satisfies $\dim \mathcal{O} = N\text{-}\dim \Gamma_{\mathrm{W}} \geqslant 1$, we call the sublattice $\Gamma_{\mathrm{W}} \oplus \mathcal{O}$ the *Kneser shape* of Γ, and one has the following embeddings of lattices:

$$\Gamma_{\mathrm{W}} \oplus \mathcal{O} \subseteq \Gamma \subseteq \Gamma^* \subseteq \Gamma_{\mathrm{W}}^* \oplus \mathcal{O}^* \tag{5.2}$$

where the asterisk denotes the dual of a lattice, as explained in Section 2.

It can be shown[6] that, for *indecomposable* CQHLs $(\Gamma, \mathbf{Q})$, Γ_{W} does *not* contain any I_l and Γ_{E_8} sublattices. In the following we will concentrate on indecomposable CQHLs, or, correspondingly, on "elementary" chiral QH fluids.

Theorem 5.1. Let $(\Gamma, \mathbf{Q})$ be an indecomposable CQHL with $\sigma_{\mathrm{H}} < 2$. Then $\mathbf{Q}$ is orthogonal to Γ_{W}, i.e., $\mathbf{Q} \in \mathcal{O}^*$, and $\Gamma_{\mathrm{W}} \subseteq \Gamma_0$, where Γ_0 is the neutral sublattice of $(\Gamma, \mathbf{Q})$. Moreover, if $\Gamma_{\mathrm{W}} \neq \varnothing$, all the inclusions in (5.2) are proper.

For a proof of this theorem and more details on the constructions above—which constitute the basis of the complete classification program of (general) CQHLs—see ref. 6, Section 6.

Theorem 5.1 has an interesting corollary concerning the *symmetry properties* of the chiral QH fluid corresponding to $(\Gamma, \mathbf{Q})$. Note that to every point in Γ there corresponds a vertex operator of the algebra of edge currents. Let $\mathscr{G}$ denote the Lie algebra—a direct sum of simple algebras A_n, D_n, and $E_{6,7}$—whose root lattice is given by the Witt sublattice Γ_W of $(\Gamma, \mathbf{Q})$. It is not hard to show[3,6] that the algebra generated by the vertex operators corresponding to the Witt sublattice Γ_W of Γ and the neutral $u(1)$-currents is the enveloping algebra of the Kac–Moody current algebra $\hat{\mathscr{G}}$ at level 1 (denoted $\hat{\mathscr{G}}_1$).

The (infinite dimensional) *symmetry algebra* $\hat{\mathscr{G}}_1$ canonically contains the (finite-dimensional) Lie algebra $\mathscr{G}$ that can be associated with global symmetry generators. Thus, the Lie group G corresponding to $\mathscr{G}$ is the *group of global symmetries* of the QH fluid. This implies that, given m electrons—fermionic quasiparticles with charge 1 and labeled by, say, $\mathbf{q}_1,..., \mathbf{q}_m \in \Gamma \subset \Gamma^*$—they transform under particular unitary irreducible representations (irreps) of G. These unitary irreps are specified as follows. Let

$$\mathbf{q}_i = \mathbf{q}_{i,W} + \mathbf{q}'_i, \qquad \text{with} \quad \mathbf{q}_{i,W} \in \Gamma_W^* \quad \text{and} \quad \mathbf{q}'_i \in \mathcal{O}^* \tag{5.3}$$

be the decomposition, according to (5.2), of the ith electron's label, $i = 1,..., m$. Then we may write $\mathbf{q}_{i,W} = \boldsymbol{\omega}_i + \mathbf{r}_i$, with $\mathbf{r}_i \in \Gamma_W$ a *root vector*, and with $\boldsymbol{\omega}_i \in \Gamma_W^*$ an *elementary weight*, i.e., a smallest length representative of the cosets (or "congruence classes" in Lie group terminology) in the quotient Γ_W^*/Γ_W (see, e.g., ref. 45). Furthermore, by the general representation theory of Lie and Kac–Moody algebras (see, e.g., ref. 46) the elementary weight $\boldsymbol{\omega}_i$ determines uniquely a unitary irrep π_{ω_i} of G according to which the one-electron state labeled by $\mathbf{q}_i$ transforms, $i = 1,..., m$.

From the general results about lattices given in ref. 43 (see also ref. 6), it follows that all the elementary weights $\boldsymbol{\omega}_i \in \Gamma_W^*$ which can appear in (5.3) are such that the corresponding irreps of $\mathscr{G}$ *can be extended* to unitary highest weight representations of $\hat{\mathscr{G}}$ at level 1. For a discussion of the latter point, see, e.g., ref. 46, Section 3.4. We will call these elementary weights "*admissible*" weights, and the ones that can occur for the simple algebras A_n, D_n, and $E_{6,7}$ are given explicitly in Appendix A.

One can show[6] that if $\dim \mathcal{O} = 1$ and $\Gamma_0 = \Gamma_W$, then all one-electron states transform under the same unitary irrep π_ω of G i.e., $\mathbf{q}_{1,W} \equiv \cdots \equiv \mathbf{q}_{m,W} \equiv \boldsymbol{\omega} \bmod \Gamma_W$.

The preceding general remarks motivate the following definition of *maximally symmetric* CQHLs.

Definition. A (proper) CQHL $(\Gamma, \mathbf{Q})$ is called *maximally symmetric* if it satisfies $\dim \mathcal{O} = 1$ and $\Gamma_0 = \Gamma_W$, i.e., the neutral sublattice of $(\Gamma, \mathbf{Q})$ and its Witt sublattice coincide. Furthermore, denoting by $\mathscr{G}$ the Lie algebra associated with the root lattice Γ_W, we require the one-electron states described by $(\Gamma, \mathbf{Q})$ to transform under a unitary irrep of $\mathscr{G}$ which can be extended to a unitary highest-weight representation of $\hat{\mathscr{G}}_1$.

Maximally symmetric CQHLs $(\Gamma, \mathbf{Q})$ are specified by the following data:

$$(L \mid {}^{\boldsymbol{\omega}}\Gamma_{\mathrm{W}}) \tag{5.4}$$

where L is an odd, positive integer, $\Gamma_{\mathrm{W}} = \Gamma_A \oplus \Gamma_D \oplus \Gamma_{E \neq 8}$ is the Witt sublattice of $(\Gamma, \mathbf{Q})$, and $\boldsymbol{\omega} \in \Gamma^*_{\mathrm{W}}/\Gamma_{\mathrm{W}}$ is an admissible weight labeling an irrep of the Lie algebra $\mathscr{G}$ associated with Γ_{W}. The possible weights $\boldsymbol{\omega}$ are further restricted by the value of L, namely $\langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle < L$ [see (5.6) below].

We note that if the Witt sublattice is a direct sum of simple root lattices, $\Gamma_{\mathrm{W}} = \Gamma_{\mathrm{W}_1} \oplus \cdots \oplus \Gamma_{\mathrm{W}_k}$, $k \geqslant 2$, then the associated Lie algebra $\mathscr{G}$ is semisimple with decomposition $\mathscr{G} = \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_k$, and correspondingly the admissible weight reads $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \cdots + \boldsymbol{\omega}_k$ where $\boldsymbol{\omega}_i \in \Gamma^*_{\mathrm{W}_i}/\Gamma_{\mathrm{W}_i}$, $i = 1, \ldots, k$. In order to get an *indecomposable* lattice, every projection $\boldsymbol{\omega}_i$ must represent a nontrivial coset in $\Gamma^*_{\mathrm{W}_i}/\Gamma_{\mathrm{W}_i}$. This can also be shown to be sufficient.[42] In the sequel, we always assume that admissible weights $\boldsymbol{\omega}$ fulfill this requirement. Hence, *all* the maximally symmetric CQHLs given in this paper are *indecomposable.*

Equivalently to (5.4), we can also specify maximally symmetric CQHLs $(\Gamma, \mathbf{Q})$ by their corresponding data pair $(K, \underset{\sim}{Q})$, once a basis has been chosen in Γ; see the beginning of Section 3. Relative to a suitable "normal" basis $\{\mathbf{q}, \mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ of Γ, $(\Gamma, \mathbf{Q})$ is specified by

$$K = \left.\left(\begin{array}{c|c} L & \underset{\rightarrow}{\omega} \\ \hline \underset{\rightarrow}{\omega}^T & C(\Gamma_{\mathrm{W}}) \end{array} \right)\right\} N \qquad \text{and} \qquad \underset{\sim}{Q} = (\underbrace{1, 0, \ldots, 0}_{N}) \tag{5.5}$$

where $L = \langle \mathbf{q}, \mathbf{q} \rangle$ is the same odd integer as in (5.4), $C(\Gamma_{\mathrm{W}})$ is the Gram matrix of the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{N-1}\}$ of Γ_{W}—in the normal basis chosen here, it equals the *Cartan matrix* of the Lie algebra $\mathscr{G}$ associated with Γ_{W}—and finally, $\underset{\rightarrow}{\omega} = (\omega_1, \ldots, \omega_{N-1})$ is the vector of the dual components of $\boldsymbol{\omega}$ which are given by $\omega_j = \langle \boldsymbol{\omega}, \mathbf{e}_j \rangle$, $j = 1, \ldots, N-1$. According to the decomposition (5.2), the basis vector $\mathbf{q}$ can be written as $\mathbf{q} = \sigma_{\mathrm{H}}^{-1} \mathbf{Q} + \boldsymbol{\omega}$.

If Γ_{W} is a direct sum of simple root lattices then $C(\Gamma_{\mathrm{W}}) = C(\Gamma_{\mathrm{W}_1}) \oplus \cdots \oplus C(\Gamma_{\mathrm{W}_k})$ is a block-diagonal matrix, and $\underset{\rightarrow}{\omega} = (\underset{\rightarrow}{\omega}_1, \ldots, \underset{\rightarrow}{\omega}_k)$. An example of data pairs (5.4) and (5.5) has been given by (4.16) and (3.12), respectively. The explicit forms of the Cartan matrices for the simple algebras A_n, D_n, and $E_{6,7}$ and of the dual vectors $\underset{\rightarrow}{\omega}$ for the admissible weights $\boldsymbol{\omega}$ are given in Appendix A.

We denote by $\Delta(\Gamma_W)$ the discriminant of the Witt sublattice Γ_W of Γ, i.e., $\Delta(\Gamma_W) := \det C(\Gamma_W) = |\Gamma_W^*/\Gamma_W|$. From (5.5), it immediately follows that for maximally symmetric CQHLs

$$\begin{aligned}\Delta = \det K &= \Delta(\Gamma_W)[L - \underset{\to}{\omega} \cdot C(\Gamma_W)^{-1}\underset{\to}{\omega}^T] \\ &= \Delta(\Gamma_W)[L - \langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle] \quad (>0)\end{aligned} \tag{5.6}$$

and

$$\sigma_H = \langle \mathbf{Q}, \mathbf{Q} \rangle = \underset{\to}{Q} \cdot K^{-1}\underset{\to}{Q}^T = \frac{\Delta(\Gamma_W)}{\Delta} > \frac{1}{L} \tag{5.7}$$

These two equations are basic for proving the following theorem.

Theorem 5.2. The symbol of a maximally symmetric CQHL $(\Gamma, \mathbf{Q})$ specified by (5.4) or, equivalently by (5.5) takes the form

$${}_{1+\text{rank}\,\Gamma_W = N}\left(\sigma_H = \frac{1}{L - \langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle}\right)^{g = \Delta(\Gamma_W)/h_{\boldsymbol{\omega}}}_{\lambda = h_{\boldsymbol{\omega}}/n_{11}} \tag{5.8}$$

where $h_{\boldsymbol{\omega}}$ is the order of the elementary weight $\boldsymbol{\omega}$ in Γ_W^*/Γ_W. Furthermore, for the relative-angular-momentum invariants $l_{\max}$ and $l_{\min}$ the equalities $l_{\min} = l_{\max} = L$ hold.

If Γ_W is a direct sum of simple root lattices, $\Gamma_W = \Gamma_{W_1} \oplus \cdots \oplus \Gamma_{W_k}$, $k \geqslant 2$, and $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \cdots + \boldsymbol{\omega}_k$, as above, then the following identities hold:

$$\text{rank}\,\Gamma_W = \sum_{i=1}^{k} \text{rank}\,\Gamma_{W_i}$$

$$\langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle = \sum_{i=1}^{k} \langle \boldsymbol{\omega}_i, \boldsymbol{\omega}_i \rangle$$

$$\Delta(\Gamma_W) = \det C(\Gamma_W) = \prod_{i=1}^{k} \det C(\Gamma_{W_i})$$

and

$$h_{\boldsymbol{\omega}} = \text{lcm}(h_{\boldsymbol{\omega}_1}, \ldots, h_{\boldsymbol{\omega}_k}) \tag{5.9}$$

where the *least common multiple* (lcm) of two integers a and b is defined by $\text{lcm}(a, b) := ab/\gcd(a, b)$, and similarly for more than two integers.

For the simple Lie algebras $A_{m-1} = su(m)$, $D_{m+2} = so(2m+4)$, $m = 2, 3, \ldots$, and $E_{6,7}$, all the ranks and determinants of their Cartan matrices, as well as all the lengths squared and orders of their admissible weights are collected in Appendix A.

Classification. Exploiting the results of Theorem 5.2 and the identities in (5.9), it is possible to list all maximally symmetric CQHLs which have a fixed value of L and whose Hall fractions σ_{H} [$>1/L$; see (5.7)] belong to a given interval. In Appendix B *all* maximally symmetric CQHLs with $l_{\min}=l_{\max}=L=3$ and $\sigma_{\mathrm{H}}<1$ are listed. They are organized in four infinite one-parameter, one infinite two-parameter, and six finite series of CQHLs. For a physically relevant subset of Hall fractions ($d_{\mathrm{H}}<21$ and odd) the resulting CQHLs are indicated in Fig. 2, and a detailed discussion is given presently.

Before entering this discussion, however, we state the most powerful implication of these results. Recalling the discussion about the shift maps in the second part of Section 4, we obtain the following *classification result*:

All L-minimal, maximally symmetric CQHLs are classified by combining the series (B1)–(B11) *given in Appendix B with Theorem* 4.7 *of Section* 4.

Here is a summary of the results given in Appendix B—the classification of L-minimal, maximally symmetric CQHLs with $1/l_{\max}=1/3<\sigma_{\mathrm{H}}<1$:

In the window $\Sigma_1^+=[1/3, 1/2)$ we find the infinite series (B1) of CQHLs with Hall fractions $\sigma_{\mathrm{H}}=N/(2N+1)$, $N=1,2,...,$ converging toward 1/2. This "basic" A- (or $su(N)$-] series needs no further explanation since it coincides with the set $\mathscr{H}_p^+$ of Theorem 4.8, which was discussed in detail at the end of the previous section.

In the "complementary" window $\Sigma_p^-=[1/2, 1)$, the classification leads to new, physically interesting perspectives.

First, in Table II we collect the symbols, as defined in (3.2), of all L-minimal, maximally symmetric CQHLs with Hall fractions σ_{H} in the subinterval $[1/2, 2/3)$. There are infinitely many such lattices with Hall fractions accumulating at 2/3.

Table II. Symbols of all L-Minimal ($l_{\max}=3$), Maximally Symmetric CQHLs with $\sigma_{\mathrm{H}}\in[1/2, 2/3)\subset\Sigma_1^-$ [a]

In $[1/2, 3/5)$[b]

$_{(n\text{-}p2)}\bullet\ {}_{3,4}(\tfrac{1}{2})^{2}_{2}$ $\bullet\ {}_{4}(\tfrac{6}{11})^{1}_{1}$ $\bullet\ {}_{5}(\tfrac{5}{9})^{1}_{1}$ $\bullet\ {}_{5}(\tfrac{4}{7})^{2}_{1},\ {}_{6}(\tfrac{4}{7})^{1}_{1}$ $\cdot\ {}_{6}(\tfrac{10}{17})^{1}_{1}$

In $[3/5, 2/3)$[c]

$_{n\text{-}p}\bullet\ {}_{5}(\tfrac{3}{5})^{3}_{1},\ {}_{6}(\tfrac{3}{5})^{2}_{1},\ {}_{7}(\tfrac{3}{5})^{1}_{1},\ {}_{7}(\tfrac{3}{5})^{2}_{2}$ ${}_{8}(\tfrac{14}{23})^{1}_{1}$ $\bullet\ {}_{9}(\tfrac{8}{13})^{2}_{1}$ ${}_{6}(\tfrac{12}{19})^{1}_{1}$ $\circ\ {}_{7}(\tfrac{7}{11})^{1}_{1}$ ${}_{7}(\tfrac{15}{23})^{1}_{1}$

and ${}_{n+1}(\tfrac{2n}{2n+2})^{\varepsilon}_{k}$ with $n=9, 10,...$

[a] Notation as in Fig. 1, with the addition that indicates a Hall fraction that has been observed in two-layer/component systems.

[b] See (B2), (B3), (4), and (B7) in Appendix B.

[c] See (B3), (B4), and (B8) in Appendix B.

From the first row in Table II we conclude that in the subinterval $[1/2, 3/5)$ *no* other fractions than 1/2, 6/11, 5/9, 4/7, and 10/17 are realized by L-minimal, maximally symmetric CQHLs! This result leads to the following significant observation:

Taking also into account the classification of generic (not necessarily maximally symmetric), low-dimensional CQHLs given in the next section (see Table VII in Appendix C, where a three-dimensional, generic, L-minimal CQHL with $\sigma_{\mathrm{H}} = 7/13$ is given), we conclude that for single-layer/component QH fluids with $\sigma_{\mathrm{H}} = N/(2N-1)$, where $N = 8, 9, \ldots$, the "*charge-conjugation* (*or particle-hole symmetry*) *picture*" provides the only "natural" theoretical description. This picture corresponds to the nonchiral decomposition $\Sigma_{\mathrm{I}}^{-} \ni \sigma_{\mathrm{H}} = 1 - \sigma'_{\mathrm{H}}$, with $\sigma'_{\mathrm{H}} < 1/2$; see refs. 29 and 30 and Appendix E.

In general, for the "second main series" of Hall fractions $\sigma_{\mathrm{H}} = N/(2N-1)$, $N = 2, 3, \ldots$, the charge-conjugation picture amounts to a description in terms of the following *charge-conjugated* A- (or $su(N)$-] *QH lattices*. These QH lattices are *composites* of two CQHLs of *opposite* chirality, meaning that they describe QH fluids which consist of electron- *and* hole-rich subfluids; see (2.1). More specifically, writing $\sigma_{\mathrm{H}} = 1 - \sigma'_{\mathrm{H}}$, we have that the charge-conjugated A-QH lattices are composites of the standard CQHL for the integer QH effect at $\sigma_{\mathrm{H}} = 1$ [see example (a) at the end of Section 2] and an *L-minimal* ($l_{\max} = 3$) CQHL corresponding to an elementary A-fluid with $\sigma'_{\mathrm{H}} = N/(2N+1) < 1/2$. Note that, given the uniqueness result (Theorem 4.8 of the previous section) for the elementary A- [or $su(N)$-] fluids in Σ_{I}^{+}, the "charge-conjugated A-fluids" with $\sigma_{\mathrm{H}} = 1 - N/(2N+1) = (N+1)/(2N+1)$ acquire a correspondingly *unique* status among all the QH fluids in Σ_{I}^{-} that are proposed by the charge-conjugation picture. Furthermore, it is shown in point (a) of Appendix E that the charge-conjugated A-fluids at $\sigma_{\mathrm{H}} = N/(2N-1)$ coincide with the "hierarchy fluids"[29, 30] at these fractions.

Contrary to the situation for the higher denominator ($d_{\mathrm{H}} > 15$) fractions of the second main series, we emphasize that for the fractions $\sigma_{\mathrm{H}} = N/(2N-1)$ with $N = 2, 3, \ldots, 7$, Tables II, III, and VII show that there *are* strictly chiral alternatives to the charge-conjugated A-fluids (see also the discussion of the "E-series" ref. 6, Section 7.4). Correspondingly, it is one of our basic contentions in this paper that *in* Σ_{I}^{-}, *the charge-conjugation picture should not be applied without further thought. For many fractions, there are chiral alternatives; see Fig.* 2. Actually, as will be discussed in Section 7, the QH physics at many of the fractions $\sigma_{\mathrm{H}} \in \Sigma_{\mathrm{I}}^{-}$ turns out to be *very complex*!

It should be emphasized that nonchiral, composite QH fluids are expected to exhibit a clear experimental signal distinguishing them from purely chiral fluids. In *nonchiral* fluids it should be possible to observe excitations of *both* chiralities at the edge of the samples, while this is in principle impossible in chiral fluids. Hence, the experiments reported in ref. 47, which do *not* find edge excitations of both chiralities at $\sigma_H = 2/3$ in the samples considered, are most interesting, and further experimentation in this direction would clearly help to deepen the understanding of the QH effect.

Next we remark that discussions and tables analogous to those for the subinterval $[1/2, 2/3)$ can be repeated for all subintervals $[(n-1)/n, n/(n+1)) \subset \Sigma_1^-$, $n = 3, 4, \ldots$. In each of these subintervals there is an infinite number of L-minimal, maximally symmetric CQHLs with Hall fractions accumulating at $n/(n+1)$.

Rather than repeating the discussions, we summarize in Table III the most relevant results for the remaining interval $[2/3, 1)$. For this interval, we present all L-minimal, maximally symmetric CQHLs of low dimension, say, $N < 7$. This restriction is motivated by our heuristic stability principle ("the smaller N and l_{max}, the more stable the corresponding QH fluid").

From Tables II and III and from our heuristic stability principle we are led to *predict* the existence of chiral QH fluids at Hall fractions 10/17, 10/13, and 12/19. Taking the symmetry structures of the corresponding maximally symmetric CQHLs into account, the fraction 10/17 is clearly predicted to be the most likely, next candidate to be observed in single-layer systems. By (B4), the one-electron states of the corresponding QH fluid are transforming under the fundamental representations of $SU(2) \times SU(5)$. Note also that, in the charge-conjugation picture 10/17 would be "conjugated" to 7/17, at which fraction there is, however, *neither* an L-minimal, maximally symmetric *nor* a generic, low-dimensional (see next section) CQHL. This conclusion is interesting, since there are some tentative experimental results suggesting the formation of a QH fluid at the fraction 10/17,[19] and there is no indication of a QH fluid at the "conjugated" fraction $\sigma_H = 7/17$.

Furthermore, comparing the data of Table I to those of Tables II and III, one immediately notices a striking *qualitative difference* between the "complementary" windows Σ_p^+ and Σ_p^-, $p = 1, 2, \ldots$. By Theorem 4.8, we have that if a Hall fraction in the windows Σ_p^+ is realized by an L-minimal,

Table III. Symbols of all L-Minimal ($l_{max} = 3$), Maximally Symmetric CQHLs with $\sigma_H \in [2/3, 1) \subset \Sigma_1^-$ and Low Dimensions $N \leqslant 6$[a]

$_{B,n\text{-}p}\bullet\ {}_{4,5,6}\left(\frac{2}{3}\right)_1^4,\ {}_6\left(\frac{2}{3}\right)_1^3$	${}_{5,6}\left(\frac{3}{4}\right)_2^2$	${}_6\left(\frac{10}{13}\right)_1^1$	$\bullet\ {}_6\left(\frac{4}{5}\right)_1^4$	$_{(2)}\circ\ {}_6\left(\frac{6}{7}\right)_1^3$

[a] Notation as in Tables I and II. See (5), (B6), (B6), and (B11) in Appendix B.

maximally symmetric CQHL, then it is *unique*. On the other hand, in the windows Σ_p^- one often finds *several structurally different* lattices realizing a given fraction. The CQHLs having the same Hall fractions are typically embedded into one another. This will be explained in more detail in Section 7 when we discuss the possibility of "structural phase transitions" in QH fluids.

The status of *even-denominator* Hall fractions will be discussed in Section 7 when the classification of generic, low-dimensional CQHLs that we present in the next section is available.

In conclusion, we note that, except for the single fraction 4/11, *all* experimentally observed Hall fractions given in Fig. 1 can be realized by either an L-minimal, maximally symmetric CQHL or a charge-conjugated A-QH lattice. All these CQHLs are of reasonably low dimension N; as a matter of fact, we have $N \leqslant 9$, except for 8/11, where the lowest dimensional L-minimal, maximally symmetric CQHL has $N = 11$.

However, before jumping to conclusions about the role of maximal symmetry in the classification of physically relevant CQHLs, we need to find a way of going at least one step beyond the classification of maximally symmetric CQHLs and see how the resulting data compare with experimental results. Such a step will be carried out in the following section by addressing the classification problem of generic CQHLs in low dimensions ($N \leqslant 4$). Just to mention two results: we shall find, e.g., at $\sigma_H = 8/11$ a non-maximally symmetric CQHL in four dimensions which is L-minimal and exhibits an $SU(2)$-symmetry; see Table IX in Appendix C. Clearly, in describing the QH fluid forming at 8/11, this CQHL competes with the 11--dimensional, maximally symmetric one mentioned above. Furthermore, the "simplest" non-L-minimal CQHL forms in dimension $N = 2$ just at the "missing" fraction $\sigma_H = 4/11$; see Table VI in Appendix C. It coincides with the proposal in the "hierarchy schemes";[29, 30] see Appendix E.

6. CLASSIFICATION OF LOW-DIMENSIONAL CQHLS

In this section we venture a step beyond the classification of maximally symmetric CQHLs presented in the last section. We provide systematic classification results for low-dimensional CQHLs that are neither necessarily L-minimal nor necessarily maximally symmetric. This allows us to get a better understanding of the role played by these two properties in the classification of physically relevant CQHLs. In the second part of this section we use our results and the phenomenological data summarized in Fig. 1 to argue that the assumption of L-minimality for physically relevant CQHLs is experimentally corroborated. The maximally symmetric CQHLs turn out to be most relevant in the windows Σ_p^+, where they are unique in the sense of Theorem 4.8. In the "complementary" windows Σ_p^- they are typically competing with generic, low-dimensional, L-minimal CQHLs. The latter ones, however, often exhibit a form of "partial" symmetry, and are in most cases contained as QH sublattices (see Section 7) in maximally symmetric CQHLs of higher dimensions.

Classification. We start by stating the precise classification results and then sketch their derivation. We have constructed the following sets of indecomposable, low-dimensional CQHLs, and correspondingly of possible "elementary" chiral QH fluids:

(N1) All one-dimensional CQHLs (they correspond to the Laughlin fluids as described at the end of Section 2).

(N2) All indecomposable CQHLs in dimension $N=2$ (e.g., for $3 \leqslant l_{\min} \leqslant l_{\max} \leqslant 7$ there are 42 such lattices).

(N3.1) All indecomposable CQHLs in dimension $N=3$, with $l_{\min} = l_{\max} = 3$ (19 CQHLs).

(N3.2) All indecomposable CQHLs in dimension $N=3$, with $3 \leqslant l_{\min} \leqslant l_{\max} = 5$, and $\sigma_{\mathrm{H}} \leqslant 3$ (191 CQHLs).

(N4) All indecomposable CQHLs in dimension $N=4$, with $l_{\min} = l_{\max} = 3$, and $\sigma_{\mathrm{H}} \leqslant 1$ (26 CQHLs).

The explicit data characterizing the CQHLs of the sets (N1)–(N4) are summarized in Tables VII–X of Appendix C.

We recall that, by definition, $l_{\min} = L_{\min}$ and $l_{\max} = L_{\max}$ for indecomposable CQHLs; see point 5 in Section 3. Moreover, given the sets (N1)–(N4), it is straightforward combinatorics to construct all primitive (see point 7 in Section 2) CQHLs with bounds on N and $l_{\max}$ as above. We note that this construction has to be carried out in order to obtain the classification of all low-dimensional ($N \leqslant 4$), L-minimal CQHLs in the windows Σ_p, with $p \geqslant 2$, by application of the shift maps $\mathscr{S}_p$ of Section 4.

Next we turn to a brief sketch of the construction of the above sets of CQHLs.

For each of the sets (N2)–(N3.2), the construction is carried out in three steps: (i) One classifies all indecomposable, integral, Euclidean lattices Γ with discriminants Δ bounded by $l_{\max}^N$; see (4.1). (ii) In the dual Γ^* of each lattice one carries out an exhaustive search for odd, primitive vectors ($\mathbf{Q}$-vectors). All $\mathbf{Q}$-vectors which belong to the same orbit under the action of the corresponding lattice automorphism group are identified, since they give rise to equivalent CQHLs; see (3.1). (iii) One has to calculate, for each resulting CQHL $(\Gamma, \mathbf{Q})$ the value of $l_{\max}$ and retain only those CQHLs satisfying the respective bounds on $l_{\max}$.

We remark that since the first step presents a highly nontrivial unsolved mathematical problem when Δ and N are getting large, the program above is bound to work only in low dimensions. Actually, we have only been able to carry it out in two and three dimensions. Specifically, the indecomposable, integral, Euclidean lattices with $N=2$ have been classified by Gauss; see, e.g., ref. 44, especially Chapter 15. In three dimensions, the very detailed discussion of "reduced forms" for the corresponding lattices by Dickson,[48] (especially the tables in Chapter 11) make a computer implementation for classifying all such lattices with, say, $\Delta < 5^3 = 125$ straightforward. Further interesting mathematical considerations related to this fist step can be found in ref. 39.

The second step is easily realized for two-dimensional lattices. Again in three dimensions the work in ref. 48 is most helpful, since it provides precise algorithms for determining the automorphism group of a given lattice. Given these algorithms and the bounds in (4.5), it is straightforward to find a computer implementation of a search routine for orbits of $\mathbf{Q}$-vectors.

The third step is tedious but computationally straightforward. The main work is to find all charge-1 vectors in Γ, which then have to be combined to form all possible symmetric bases needed in order to calculate $l_{\max}$; see (3.6).

For a better organization of the CQHLs in (N3.2), it is convenient to introduce another relative-angular-momentum invariant: Similarly to (3.6), we denote by $o\mathscr{B}_{\mathbf{Q}}$ the set of all ordered, symmetric bases of Γ, $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$, i.e., $\langle \mathbf{Q}, \mathbf{q}_i \rangle = 1$, for $i = 1, 2, 3$, and $\langle \mathbf{q}_1, \mathbf{q}_1 \rangle \leqslant \langle \mathbf{q}_2, \mathbf{q}_2 \rangle \leqslant \langle \mathbf{q}_3, \mathbf{q}_3 \rangle$. Then one can show that for all lattices considered in (N3.2) the following invariant is welldefined:

$$l_2 := \min_{\substack{\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} \in o\mathscr{B}_{\mathbf{Q}} \\ \langle \mathbf{q}_1, \mathbf{q}_1 \rangle = l_{\min},\ \langle \mathbf{q}_3, \mathbf{q}_3 \rangle = l_{\max}}} \langle \mathbf{q}_2, \mathbf{q}_2 \rangle \tag{6.1}$$

and its possible values are 3 and 5. The set (N3.2) can be split into three subsets characterized by $[l_{\min}, l_2, l_{\max}] = [3, 3, 5]$, $[3, 5, 5]$, and $[5, 5, 5]$, respectively. The corresponding compilations of CQHLs are summarized in Table VIII of Appendix C. Clearly, the subset with invariants $[5, 5, 5]$ contains all the (indecomposable) images under the shift map $\mathscr{S}_1$ of the CQHLs listed in set (N3.1); the corresponding inverse images are indicated in Table VIII.

In order to obtain the set (N4) we have applied the following procedure. Making use of the special form that the data pairs (K, Q) characterizing these CQHLs take in suitable symmetric bases [see (C.3) in Appendix C], the positivity of K implies that all six coefficients, $a_1, a_2, \ldots, c$

necessarily have an absolute value which is strictly less than three. Based on this observation a simple computer routine can be used to generate the data pairs $(K, \underset{\sim}{Q})$ (relative to symmetric bases) of all CQHLs which belong to the set (N4). Identifying all the data pairs which are related by a mere change of basis in an underlying CQHL [see (3.1)] and checking for indecomposability, one obtains the result summarized in Table IX. Actually, the indecomposability of lattices with discriminant $\Delta \leqslant 25$ could be checked by comparison with the classification results given in ref. 43. The lattices with discriminants Δ exceeding 25 had to be considered case by case.

This completes the description of our procedures for obtaining the sets (N1)–(N4). Next we shall see what these results imply with respect to the role played by L-minimality and maximal symmetry in the classification of physically relevant CQHLs.

***L*-Minimality and Maximal Symmetry vs. Experiment.** We first recall that an L-minimal CQHL with $\sigma_{\mathrm{H}} \in \Sigma_p = [1/(2p+1), 1/(2p-1))$ is characterized by its primitivity (see point 7 in Section 2) and the equalities $L_{\min} = l_{\min} = L_{\max} = l_{\max} = 2p+1$, $p = 1, 2,\ldots$ Given the explicit data in Appendices B and C, we can ask: Which Hall fractions σ_{H}, e.g. in the window Σ_1 are "strongly non-L-minimal?" Here, *strongly non-L-minimal* means that these fractions *can* be realized by a non-L-minimal (indecomposable or composite) CQHL with $N \leqslant 3$, but *neither* by a low-dimensional ($N \leqslant 4$), L-minimal CQHL *nor* by a maximally symmetric one of *arbitrary* dimension. Besides this "strong" form of non-L-minimality we may also define a "weaker" form. Let us call a Hall fraction *weakly non-L-minimal* if it can be realized by a non-L-minimal CQHL with $N \leqslant 3$ and if there is also a maximally symmetric, L-minimal realization, however, *only* in higher dimensions, say, with $N \geqslant 10$. Recall the phenomenological discussion at the end of the last section, where $N \simeq 10$ has been argued to provide an approximate, heuristic upper bound on the dimension of maximally symmetric CQHLs which are physically relevant.

A compilation of strongly and weakly non-L-minimal Hall fractions is given in Table IV. The non-L-minimal (indecomposable or composite) CQHLs realizing these fractions are indicated by the values of their invariants $l_{\min}$, l_2, and $l_{\max}$, respectively, and the corresponding explicit data pairs $(K, \underset{\sim}{Q})$ can be found in Tables VI and VIII of Appendix C. In Table IV the dimensions in which maximally symmetric lattices exist for the weakly non-L-minimal fractions are indicated in brackets. All other notations are as in Table I.

Upon closer inspection, Table IV is most revealing. The "simplest" strongly non-L-minimal situations are encountered at $\sigma_{\mathrm{H}} = 4/11$ and $8/15$.

Table IV. Strongly and Weakly Non-L-Minimal Hall Fractions σ_H in the Window $\Sigma_1 = [1/3, 1)$[a]

$\cdot\,\frac{4}{11}$	$\frac{7}{19}$	$\frac{3}{8}$	$\frac{5}{13}$	$\frac{7}{17}$	$\circ\,\frac{8}{15} = \frac{1}{3} + \frac{1}{5}$	$\frac{11}{19}$
[3, 5]	[3, 5, 5]	[3, 5, 5]	[3, 5, 5]	[3, 3, 5]	$[3] \oplus [5]$	[5, 5, 5]
[5, 5, 5]	[5, 5, 5]			[5, 5, 5]	$[5, 5] \oplus [5]$	
$\frac{13}{21} = \frac{1}{3} + \frac{2}{7}$	...					
$[3] \oplus [5, 5]$						
$\frac{9}{11}$	$\frac{13}{15} = \frac{1}{3} + \frac{1}{3} + \frac{1}{5}$					
[3, 5, 5]	$[3] \oplus [3] \oplus [5]$	[3, 5, 5]	...			
$(N \geqslant 17)$	$(N \geqslant 25)$	$(N \geqslant 33)$				

[a] Notation explained in the text.

For both fractions there is a *two*-dimensional [3,5]-CQHL with invariants $\lambda = g = 1$. It is indecomposable in the first, and composite in the second case. As a matter of fact, we note that the latter situation provides one of the "simplest" examples of a *composite chiral* QH fluid, namely a composite of two basic Laughlin fluids. Clearly, at $\sigma_H = 8/15$ the description in the charge-conjugation picture, $8/15 = 1 - 7/15$ [where the 7/15 hole subfluid is described by the unique L-minimal CQHL $(3|{}^1A_6)$ in dimension $N = 7$; see the discussion in Section 5] competes with the above non-L-minimal solution. Applying the results of Appendix E, the above [3, 5]-CQHL at $\sigma_H = 4/11$ corresponds to the QH fluids predicted by the Haldane–Halperin (HH)[29] and Jain–Goldman (JG)[30] hierarchy schemes at "level" two and three respectively.

Experimentally, there seems to be only very weak support for a QH fluid at $\sigma_H = 4/11$ (see ref. 20 and ref.12 therein), and some first indications of the Hall effect at 8/15 have only been found recently in very high quality samples.[17, 18] Apparently, the formation of QH fluids at these two fractions is a very delicate matter.

More surprisingly, there is a persistent absence of experimental indications of the QH effect at the non-L-minimal fractions 7/19, 5/13(!), 7/17(!), 11/19, 13/21, 9/11(!), 13/15(!), and 17/19. The fractions marked with "(!)" are well separated from experimentally strong fractions nearby and thus *a priori* they are expected to be experimentally observable. This should be further confronted with the fact that none of the fractions in Σ_1 which are realizable by L-minimal CQHLs with $N \leqslant 3$ is lacking experimental observation! We note that, in the two hierarchy schemes, fluids at *low*(!) "levels" are predicted at all these fractions. In the HH picture, there are, at all fractions above, fluids at "level" 3, with the exception of 11/19 and 13/21, where fluids form at "level" 5. In the JG scheme, the corresponding fluids are found at "level" 2, except for the last three fractions where they form at "level" 3, 4, and 5, respectively. From the point of view of QH lattices all "hierarchy fluids" predicted at the fractions above are *non-Euclidean* with the exception of those at 7/17 and 7/19; see Appendix E. In these two cases, they coincide with our non-L-minimal proposals with $l_{\min} = 3$ and $l_{\max} = 5$, respectively, listed in Table IV.

Recalling the heuristic stability principle of Section 4 (we have that the observations above lead to the following):

Strong Stability Principle. The most stable chiral QH fluids are described by L-minimal CQHLs, and the smaller the lattice dimension N, the greater the stability of the corresponding fluid.

This heuristic stability principle, with the prominence of L-minimal CQHLs implied by it, is rather pleasing in the light of Theorem 4.7, which states that all sets $\mathscr{H}_p$ of L-minimal, primitive CQHLs in the windows Σ_p, $p = 2, 3,\ldots$, stand in one-to-one correspondence with $\mathscr{H}_1$ in Σ_1.

Furthermore, given the stability principle above and the result of Theorem 4.8, it would appear to be justified to claim that there is now a firm understanding of the "structural organization" of QH fluids in the windows Σ_p^+, $p = 0, 1, 2,\ldots$ We note that, in particular, at the Hall fractions $\sigma_{\mathrm{H}} = N/(2pN+1)$, $N = 1, 2,\ldots$, which belong to the windows Σ_p^+, the HH-hierarchy picture,[29] the JG picture,[30] and our "L-minimal CQHL picture" are equivalent. For details, see Appendix E.

Combining the two preceding remarks, we conclude that the challenging ground for deepening the understanding of the QH effect lies in the "complementary" windows Σ_p^-, $p = 1, 2,\ldots$, and in particular in the "fundamental domain" $\Sigma_p^- = [1/2, 1)$. In this window, room is found for an interesting competition between three classes of L-minimal CQHLs: (i) the generic, low-dimensional ($N \leqslant 4$) CQHLs with no symmetry restrictions on their structure, (ii) the class of maximally symmetric CQHLs of fairly low dimensions (typically $N \leqslant 9$), and (iii) the (nonchiral) charge-conjugated A-QH lattices discussed in Section 5. This competition and its consequences, such as the prediction of possible "structural phase transitions," appears to be missed in the hierarchy schemes. It is one of the main issues we address in our final section.

7. SUMMARY AND PHYSICAL IMPLICATIONS OF THE CLASSIFICATION RESULTS

In this final section the key insights and conclusions of the previous sections are summarized and completed. In particular, the status of the two main restrictions assumed in our classification, chirality and *L-minimality*, is discussed in detail. Several new experiments that could help to further deepen the understanding of the QH effect, in particular, of the "structural organization" of QH fluids, are proposed.

Stability Principles. Based on the physical meaning of the CQHL invariants N [the number of channels in the corresponding QH fluid; see (A2) in Section 2] and $l_{\max}$ [the smallest relative angular momentum of a pair of a certain type of electrons that are excited above the QH fluid's ground state; see (3.8)], we have motivated in Section 4 the heuristic stability principle that *the smaller the invariants N and $l_{\max}$ the more stable the corresponding QH fluid.*

For a sharpening of this stability principle, the introduction of the notion of L-minimality has proven to be effective. *L-minimality* says that all the minimal relative angular momenta between any two identical types of electrons excited above a QH fluid's ground state are the *same* ("homogeneity"), and that, furthermore, $l_{\max}$ assumes the *smallest possible* value ("minimality") consistent with the value of the Hall fraction σ_{H}; see below (4.6). A detailed confrontation of our classification results (summarized in Appendices B and C and discussed in Sections 5 and 6) with the experimental data summarized in Fig. 1. then leads to the following strong stability principle: *The most stable chiral QH fluids are described by L-minimal CQHLs, and the smaller the lattice dimension N, the greater the corresponding fluid's stability.*

Furthermore, experimental data on single-layer systems suggest the respective values 10 *and* 7 *as heuristic upper bounds for the invariants N and* $l_{\max}$ *of physically relevant CQHLs* (see also the discussion preceding Theorem 4.2). This observation is most powerful in combination with Theorem 4.1, which states that the set of CQHLs satisfying such bounds is *finite.*

We continue this subsection with two compilations of Hall fractions where experimental indications of QH fluids would in the first case strengthen the conclusions above and in the second case, would pose new interesting questions about the physics underlying the QH effect. For a partial summary of the subsequent results, see Fig. 2 in Section 1.

(a) *New fractions at which QH fluids can be expected to form.* Given the above stability principles, there are basically two ways to predict new Hall fractions at which one could expect the formation of QH fluids in single-layer systems from the data given in Appendices B and C.

First we shall argue for new fractions in the window $\Sigma_{\mathrm{I}} = [1/3, 1)$. There candidates are fractions that can be realized by "simple" maximally symmetric CQHLs, where "simple" means L-minimal, low-dimensional, and the Witt sublattice [which encodes the symmetry properties of the fluid; see (5.1)] is either simple or semi-simple but with at most *two* summands. The most obvious such candidates are the three fractions 10/13, 10/17, and 12/19 of Table II, and the next "member" in the basic A- [or $su(N)$-] series [see (B1)], namely 10/21. The first three fractions are realized by CQHLs in six dimensions, the latter by one in ten dimensions. All four lattices are indecomposable and have level $l=\lambda g=1$ which means that, by Theorem 4.5, a charge–statistics relation holds for them. In addition to these fractions further candidates in the window Σ_1 can be inferred from Table IX containing all indecomposable, L-minimal CQHLs in four dimensions. Here two fluids with a partial $SU(2)$ and one with a partial $SU(2)\times SU(2)$-symmetry are predicted to form at $\sigma_H=6/7, 13/17$, and 14/19, respectively. Moreover, a generic fluid exhibiting no continuous symmetries might form at $\sigma_H=11/13$.

Second, in the windows $\Sigma_p=[1/(2p+1), 1/(2p-1))$, $p=2, 3,...$, new QH fluids are predicted by acting with the shift maps $\mathscr{S}_{p-1}$ on the CQHLs corresponding to well-established fluids with $\sigma_H\in\Sigma_1$; see Section 4, in particular transformation property (4.12). The most immediate fluids whose shift map images might be considered are the ones belonging to the A-series with Hall fractions $\sigma_H=N/(2N+1)$. This leads to predictions of QH fluids at, e.g., 2/13, 4/17, and 5/21. We note that, from a QH lattice point of view, our results at the fractions $\sigma_H=N/(2N+1)$ and at their shifted images coincide with the proposals given in both the Haldane–Halperin[29] and the Jain–Goldman[30] hierarchy schemes; see Appendix E. However, at most of the other fractions the pictures can differ significantly, as we explain in detail in the remaining part of this section.

(b) *"Missing" Hall fractions.* Our considerations here are not only based on the two sets of classification results summarized in Appendices B (L-minimal, maximally symmetric CQHLs) and C [all indecomposable CQHLs with $N\leqslant 3$ (4) and $l_{\max}\leqslant 5$ (3)], but also on the investigation of the *composite* CQHLs that can be built from the ones listed there, provided their invariants N and $l_{\max}$ satisfy the respective bounds. For brevity we restrict attention to odd-denominator fractions in the window Σ_1. A general discussion of the status of even-denominator fractions will be given below.

The strongest statement we can make about "missing" fractions in Σ_1 is the following: The data mentioned above provide *no* CQHLs at the fractions 6/17, ∘ 9/17, 8/19, 10/19, 13/19, 8/21, 11/21,..., and hence, *no chiral* QH fluids are expected to form at these fractions. (When listing fractions in this section, the dots ... always indicate further fractions with $d_H>21$, and the experimental status of the fractions in single-layer systems is indicated as in Fig. 1. In other words, finding an experimental signal at one of these fractions forces us either to go beyond our classification results or to reconsider some of our basic assumptions. For example the implications for the status of the chirality assumption which follow from the experimental data at $\sigma_H = 9/17$ (and for that matter, would also result from signals at 10/19 and 11/21) are discussed in the next subsection.

By reversing the line of arguments that lead to the strong stability principle in Section 6, we can make further nontrivial predictions of "missing" fractions. Namely, assuming (i) L-minimality to be a necessary property of stable QH fluids, and (ii) that our data is exhaustive (which means, in particular, that generic L-minimal CQHLs with $N \geqslant 5$ are physically irrelevant), then *no* stable chiral QH fluid can form at the fractions 4/11, 5/13, 8/15, 7/17, 7/19, 11/19, 13/21,.... These fractions have been called strongly non-L-minimal in Section 6; see Table IV. We note that a detailed analysis of the implications resulting from the experimental indications at 4/11 and 8/15 can also be found there. (The fraction 8/15 finds a natural explanation in the charge-conjugation picture, as discussed presently, and the weak experimental data at 4/11 might indeed indicate the *only* QH fluid corresponding to a non-L-minimal CQHL, which, in this case, would be two-dimensional.) Assuming, in addition, a heuristic upper bound on the dimension N of CQHLs that can be realized physically, say $N \leqslant 10$, as mentioned above, then further "missing" fractions are predicted to be 9/11(17), 13/15(25), 17/19(33), as well as 11/17(23), 14/17(20), 16/17(18), 15/19(15), 16/19(19), 18/19(20), 16/21(19), 17/21(17), 19/21(37),.... The first three fractions in this list have been called weakly non-L-minimal and appeared in Table IV. All fractions are listed together with the dimension in which the lowest dimensional maximally symmetric, L-minimal CQHL can be found realizing that Hall fraction.

Given these predictions, it would certainly be most interesting to carry out further experimental investigations in the regions around the indicated "missing" Hall fractions. The status of some of these fractions in the hierarchy schemes has been discussed toward the end of Section 6.

Composite CQHLs and Charge Conjugation. What can we infer from experiment about the necessity to consider composite *chiral* QH lattices in the description of single-layer QH fluids? The answer is, there are *no* experimental data in Fig. 1 conveying need for composite CQHLs, *except* possibly at $\sigma_H = 2N/(2N+1)$ where direct sums of two identical (indecomposable) CQHLs from the basic A-series should not be ruled out *a priori*; see the discussion below, in the subsection about "structural phase transitions." To substantiate this claim, let us list, e.g., all Hall fractions exhibited by low-dimensional ($N \leqslant 4$), L-minimal, composite CQHLs in $\Sigma_1^- = [1/2, 1)$: $\bullet\, 2/3 = 1/3 + 1/3$, $\bullet\, 4/5 = 2/5 + 2/5$, $5/6 = 1/3 + 1/2$, $9/10 = 1/2 + 2/5$, $11/15 = 1/3 + 2/5$, $14/15 = 1/3 + 3/5$, $16/21 = 1/3 + 3/7$,.... We note that all such composite lattices necessarily have $\sigma_H \geqslant 2/3$. The claim can be further corroborated by also inspecting higher dimensional, as well as non-L-minimal, composite CQHLs.

In multi-layer/component systems with nearly independent components, e.g., with a strong suppression of tunneling between the different layers, the picture will, of course, be different, and fractions listed above might possibly arise.

The second question is whether the experimental data in Fig. 1 are suggestive of QH fluids that are composites of subfluids with *opposite* chiralities. For single-layer systems the commonly accepted charge-conjugation (or particle–hole symmetry) picture[29, 30] assumes this to be so. Actually, in this picture, the Hall physics at the fractions $\sigma_H \in \Sigma_p^- = [1/2, 1)$ is assumed to be the "charge-conjugated" mirror image, $\sigma_H = 1 - \sigma'_H$, of the one at the corresponding fractions $\sigma'_H \in (0, 1/2]$. In particular, at two "conjugated fractions" (σ_H, σ'_H) the likelihoods of formation and the stability properties of the corresponding QH fluids are expected to be approximately the same.[30] Although this picture is contained in our general framework presented in Section 2 [see (2.1) and Appendix E], we argue that it is not *in general* in accordance with the available experimental data.

Let us see more precisely what the experimental evidence for or against the charge-conjugation picture is in single-layer systems. A first look at Fig. 1 shows that there are 11 pairs of conjugated fractions (σ_H, σ'_H) where at both fractions QH fluids of similar stability have been established, and which thus are consistent with the charge-conjugation picture. These 11 pairs, however, have to be confronted with 10 (!) pairs of conjugated fractions (σ_H, σ'_H) where either only one member is observed or the stability status of the two members is markedly different. Taking a closer look at the experimental data, one realizes that 8 of the 11 pairs supporting charge conjugation are of the form $(N/(2N+1), (N+1)/(2N+1))$, i.e., they relate fractions of the basic A-series with ones belonging to the "second main experimental series."

As discussed at the end of Section 5, it is natural and in some cases necessary to take the charge-conjugation picture into account when discussing the QH physics at the fractions of the second main series, $\sigma_H = N/(2N-1)$, $N = 2, 3$,.... The particular *nonchiral, composite* QH lattices associated with these fractions in the charge-conjugation picture have been called *charge-conjugated A-QH lattices*. They have a *unique* status among all charge-conjugated QH lattices in Σ_p^-; see Section 5.

We note, however, that for the first six members (2/3 through 7/13) of the second main series there are also *strictly chiral, L-minimal alternatives* a fact that is rather interesting, in the light of the results reported in ref. 47. In the experiments reported there, one has been looking for the signature of a charge-conjugation QH fluid at $\sigma_{\rm H} = 2/3(=1-1/3)$, namely, the existence of edge excitations of *both* chiralities; see Section 2. But no evidence was found for this signature, a result that would be consistent with the proposal of a *strictly chiral* fluid at that fraction. Further physically interesting implications of *chiral* QH lattices are discussed below, in the subsection about "structural phase transitions."

There is another important observation to be made: In the realm of CQHLs there are only *non-L-minimal* CQHLs at the fractions · 4/11, 5/13, and 7/17, while at the "conjugated" values ◦ 7/11, • 8/13, and · 40/17 there are *L-minimal* (maximally symmetric) CQHLs of dimension 7, 9, and 6, respectively. Given the fact that the first three fractions are experimentally only very weakly indicated or unobserved, while the latter three are clearly observed or indicated, we favor the chiral explanations for the latter three fractions over the ones of the charge-conjugation picture.

In conclusion, we are tempted to claim that for single-layer systems the experimental data do not support the charge-conjugation picture *in general.* Since this claim may appear to remain doubtful, further experiments of the type reported in ref. 47, would be most welcome.

Status of Even-Denominator Hall Fluids. First we emphasize that in the framework adopted in the present work, the description of QH fluids at fractions with even denominators $d_{\rm H}$ is *not* an impossibility. This is satisfying since, experimentally, even-denominator QH fluids are well established at $\sigma_{\rm H} = 1/2$ in *two-layer/component* systems,[25, 26] and there are celebrated data at $\sigma_{\rm H} = 5/2$ observed in *single-layer* systems.[27, 28]

Second, theoretically, the most interesting *fact about even-denominator CQHLs is that their charge parameters λ are necessarily even*; see Theorem 4.3. Phenomenologically, this translates into the prediction that *in such fluids quasiparticles may be excited above the ground state which have (fractional) charges $e^* = 1/(\lambda d_{\rm H}) \leqslant 1/(2d_{\rm H})$(!)*; see (3.4). The even-λ observation acquires further meaning when we note that all odd-denominator QH lattices which are consistent with the above strong stability principle and the respective phenomenological bounds on N and $l_{\max}$ are characterized by $\lambda = 1$. Thus, the charge parameter λ appears to play a dichotomizing role between odd-and even-denominator QH fluids.

Third, we must ask the crucial question: Which even-denominator fractions are predicted in our framework? To be more precise, taking over (i) the strong stability principle, (ii) the experimentally supported upper bounds on the invariants N and $l_{\max}$, and (iii) the fact that, phenomenologically, there is little need for composite CQHLs, we ask:

Which even-denominator Hall fractions in Σ_1 can be realized by L-minimal, indecomposable CQHLs that are either maximally symmetric with $N \leqslant 10$, or generic with $N \leqslant 4$? The answer is surprisingly short! We give the resulting fractions and indicate in round and square brackets the dimensions of the corresponding maximally symmetric and generic CQHLs, respectively: $1/2\ [2],\ (3, 4,...),\ 3/4[4],\ [4] \supset su(3), (5, 6,...)$, $5/6(7, 8,...)$, $5/8[4] \supset su(2), (9, 10,...)$, $7/8(9, 10,...)$. The generic lattices at 1/2, 3/4, and 5/8 are given explicitly in Tables VI and IX in Appendix C, while all the maximally symmetric ones with Hall fractions $(2n-1)/(2n)$ are structurally similar. Their Witt sublattices are given by ${}^1A_{2(n-1)}{}^1A_1{}^1A_1, {}^1A_{2(n-1)}{}^2A_3,...$; see (B2) and (B5) in Appendix B and the discussion in the next subsection. Since for $n = 2, 3,...$, the Witt sublattices of the lowest-dimensional realizations are semisimple with three summands, we do not expect these lattices to present phenomenologically plausible proposals. This, in turn, leaves us, *for the window* Σ_1, *with the prediction of even-denominator QH fluids at* $\sigma_{\mathrm{H}} = 1/2, 3/4$, and 5/8.

We recall that, as mentioned in Section 1, there are convincing arguments[14] that in a *single-layer* QH system there are *no* plateaus at $\sigma_{\mathrm{H}} = 1/2, 1/4, 3/4$, (and other even-denominator fractions). The ground state of a QH system at the corresponding filling factors is argued to be a gapless Fermi liquid.

For *double-layer* (or *wide-single-quantum-well*) QH systems, however, the proposals made above are very natural. For example, at $\sigma_{\mathrm{H}} = 1/2$ we have a maximally symmetric CQHL with symbol [see (3.2)] and data [see (5.4)] given by ${}_3(1/2)^2_2(3|{}^1A_1{}^1A_1)$. This three-dimensional example has been discussed in Section 1. The two $A_1 = su(2)$ summands forming its Witt sublattice Γ_{W} make it a natural candidate for describing a QH fluid with an $SU(2)_{\mathrm{spin}}$ and an $SU(2)_{\mathrm{layer}}$ symmetry. Similar discussions can be repeated for the other even-denominator QH lattices mentioned above.

Embeddings of CQHLs and Structural Phase Transitions. A rather remarkable consequence of our study of QH lattices is that, staying in the context of *chiral* and *L-minimal* QH lattices, as motivated above, the interval of Hall fractions $0 < \sigma_{\mathrm{H}} \leqslant 1$ can naturally be organized into "windows" in a twofold way.

First, defining the windows $\Sigma_p = [1/(2p+1), 1/(2p-1))$, $p = 1, 2,...$, the characterizing property of L-minimal CQHL with $\sigma_{\mathrm{H}} \in \Sigma_p$ is that they saturate the bound $1/\sigma_{\mathrm{H}} \leqslant l_{\max}$ given in Theorem 4.2, i.e., they have $l_{\max} = 2p+1$. We recall that, by Theorem 4.7, *all the sets of L-minimal CQHLs with* $\sigma_{\mathrm{H}} \in \Sigma_p$ *are in one-to-one correspondence with one another.* These correspondences are realized by the *shift maps* discussed in Section 4 and lead to the result that, when discussing L-minimal CQHLs, we can restrict attention to the "*fundamental window*" Σ_1. We will make use of this fact in the remaining part of this subsection.

Second, each window Σ_p can be divided into two subwindows Σ_p^+ and Σ_p^- by the mid value of $1/(2p)$. The interesting fact behind this division is that the two resulting subwindows exhibit very different "*structural organization.*" While in the windows $\Sigma_p^+ = [1/(2p+1), 1/(2p))$ there are *unique* L-minimal CQHLs at the fractions $\sigma_H = N/(2pN+1)$, $N = 1, 2, \ldots$ (see Theorem 4.8), one infers from the data in Appendices B and C that in the "complementary" windows $\Sigma_p^- = [1/(2p), 1/(2p-1))$ typically *several inequivalent* CQHLs can be found at a given Hall fraction σ_H. An interesting question then is: What is the relationship between CQHLs which have the same Hall fraction? Furthermore, what does this relationship imply at the level of QH fluids? In order to answer these two questions, we introduce the concept of QH-lattice embeddings.

Definition. A QH lattice $(\Gamma', \mathbf{Q}' \in \Gamma'^*)$ is embedded into another QH lattice $(\Gamma, \mathbf{Q} \in \Gamma^*)$ if (i) both QH lattices exhibit the same Hall fraction, i.e., $\sigma'_H = \langle \mathbf{Q}', \mathbf{Q}' \rangle = \langle \mathbf{Q}, \mathbf{Q} \rangle = \sigma_H$, (ii) Γ' is a sublattice of Γ, and (iii) the two charge vectors $\mathbf{Q}'$ and $\mathbf{Q}$ are compatible in the sense that all multi-electron/hole states described by $(\Gamma', \mathbf{Q}')$ remain physical states when viewed (via the lattice embedding $\Gamma' \subset \Gamma$) as states described by $(\Gamma, \mathbf{Q})$. In particular, all the electric charges stay the same, i.e., $\langle \mathbf{Q}', \mathbf{q}' \rangle = \langle \mathbf{Q}, \mathbf{q}' \rangle$ for all $\mathbf{q}' \in \Gamma' \subset \Gamma$.

At the level of symbols [see (3.2)] we denote such embeddings by

$$ {}_{N'}\!\left(\frac{n_H}{d_H}\right)^{g'}_{\lambda'} [l'_{\min}, l'_{\max}] \hookrightarrow {}_{N}\!\left(\frac{n_H}{d_H}\right)^{g}_{\lambda} [l_{\min}, l_{\max}] \tag{7.1} $$

Note that, as an immediate consequence of definition (3.7), $l'_{\min} \geqslant l_{\min}$.

Physically, a QH fluid described by the QH lattice $(\Gamma', \mathbf{Q}')$ which is embedded into another lattice $(\Gamma, \mathbf{Q})$ is characterized by a *restricted* set of possible multi-electron/hole excitations above the ground state, as compared to the corresponding set of the fluid associated with the lattice $(\Gamma, \mathbf{Q})$. Furthermore, since the neutral sublattice [see (4.8)] of $(\Gamma', \mathbf{Q}')$ **is a sublattice of the neutral sublattice of** $(\Gamma, \mathbf{Q})$, the embedded fluid exhibits a (global) symmetry group G' [see (5.3)] which is a *subgroup* of G, the symmetry group exhibited by the fluid associated with $(\Gamma, \mathbf{Q})$. Thus, in this precise sense, *the embedded fluid exhibits a more restricted symmetry than the one it embeds into.* Put differently, *going from a QH fluid to an embedded subfluid corresponds to a "reduction or breaking of symmetries."* (As a mathematical aside, we remark that the study of embeddings of maximally symmetric CQHLs into one another is equivalent to the study of regular conformal embeddings of level 1 Kac–Moody algebras and the respective branching rules. For recent results on the latter subject, see, e.g., the references in ref. 49.) Experimentally, symmetry breaking might be realized in *phase transitions* that are driven at a given Hall fraction by varying external control parameters. Hence, it is most interesting to see at which fractions in Σ_1^- such "structural" phase transitions can be expected within our framework.

Motivated by the observations in the first two subsections above, we answer this question by taking into account the following physically relevant sets of CQHLs: (i) all generic, L-minimal CQHLs in low dimensions, $N \leqslant 4$ (see Appendix C), (ii) all maximally symmetric, L-minimal CQHLs in dimensions $N \leqslant 10$ (see Appendix B), and (iii) all composites of two identical lattices belonging to the basic A-series given in (B1) of Appendix B. The Hall fractions in Σ_1^- at which a CQHL embedding, or "chains" of CQHL embeddings, can be found are listed, together with the corresponding lattices, in Table X of Appendix D. The resulting fractions are ${}_{B,n\text{-}p}{\bullet}2/3$, ${}_{B\text{-}p}{\bullet}3/5$, ${\bullet}4/5$, ${\bullet}4/7$, ${}_{(B\text{-}p)}{\bullet}5/7$, ${}_{(2)}{\circ}6/7$, ${\bullet}5/9$, and the even-denominator fractions ${}_{(2)}{\bullet}\,1/2$ and $(2n-1)/(2n)$, with $n=2$, 3, and 4.

This result can actually be sharpened by taking the structure of the CQHLs involved into account (especially their symmetry groups). Given that at the fractions $n/(n+1)$ with $n = 3, 4, 5, 6$, and 7 already the lowest dimensional pairs of embedded CQHLs involve structurally complex Witt sublattices (with three summands and dimensions $N \geqslant 5$), we do not expect the proposals at these fractions to be phenomenologically very relevant. To summarize, *in* Σ_1^- *the Hall fractions at which structural phase transitions are likely to occur are predicted to be* ${}_{B,n\text{-}p}{\bullet}2/3$, ${}_{B\text{-}p}{\bullet}3/5$, ${\bullet}4/7$, ${}_{(B\text{-}p)}{\bullet}5/7$, ${\bullet}5/9$, *and* ${}_{(2)}{\bullet}1/2$! Confronted with the experimental data, we find it most remarkable that precisely at the three fractions 2/3, 3/5, and 5/7 at which there are low-dimensional CQHL embeddings ($N \leqslant 4$), phase transitions have been observed or are experimentally plausible. Observations of phase transitions at $\sigma_H = 4/7$ and 5/9 would, of course, further support the proposed picture of structural phase transitions. Thus, experiments are encouraged at these fractions!

One question that remains is whether *other types of phase transitions* can occur *in the windows* Σ_p^+ where we have the A-series of unique L-minimal CQHLs? The answer is *yes*! We briefly explain why. So far we have basically ignored the spin degrees of freedom in our discussion. However, a systematic incorporation of *spin phenomena* into our framework is straightforward and has been discussed in detail in ref. 5; see also ref. 6. Basically, such an extended framework for QH fluids with dynamical spin degrees of freedom incorporates (i) all the data forming a QH lattice (Γ, Q) and (ii) it additionally requires a *polarization vector* $\boldsymbol{\delta} \in \Gamma^*$. The polarization vector $\boldsymbol{\delta}$ specifies the spin polarization of the excitations in the system (relative to some given direction) similarly to the way the charge vector $\mathbf{Q}$ specifies their electric charges; see (2.11). Given, e.g., a CQHL with a (neutral) $A_1 = su(2)$ sublattice, it has been shown in ref. 5, Section 6, that such a lattice can naturally be used to describe *either* a QH fluid with a spin-singlet ground state [from which $SU(2)_{\text{spin}}$ degrees of freedom can be excited] *or* a QH fluid with a fully polarized ground state (from which only polarized quasi-particles can be excited), exhibiting, however, an internal $SU(2)$ symmetry. Datawise, the two QH fluids are only distinct by the form of their associated polarization vectors! In ref. 5, Section 7, the simplest examples of such fluids have been discussed. They form at the fractions $\sigma_H = 2/(4p+1)$, $p = 1, 2,\ldots$, and are based on the maximally symmetric, L-minimal CQHLs with data $(2p+1 \mid {}^1A_1)$; see (5.4). Experimentally, the two QH fluids, one having a spin-singlet ground state and the other one a polarized ground state with an internal symmetry, can be distinguished, in principle, by their magnetic susceptibilities and by their quantum Hall effects for the spin currents; see ref. 5, Section 7. In conclusion, *at fractions in Σ_p^+ we do not expect structural phase transitions*; *however, spin-induced phase transitions are clearly possible*! More details on this will be given elsewhere.[42]

Finally, we ask whether one should expect to observe *phase transitions at $\sigma_H = 1$*. The unique L-minimal ($l_{\max} = 1$) CQHL is the one-dimensional Laughlin lattice with $m = 1$; see example (b) in Section 2. Thus, any other CQHL realizing this fraction necessarily has to be *non-L-minimal* ($l_{\max} \geqslant 3$), a fact that suggests a *markedly reduced stability* for the corresponding fluids, as compared to the (L-minimal) Laughlin fluid! Moreover, by Theorem 4.4 we know that any other indecomposable CQHL at this fraction exhibits a charge parameter λ strictly larger than 1. By an argument similar to the one in (4.4), this leads to the prediction of *fractional* charges in these fluids! For the purpose of illustration, we give the lowest dimensional examples of such lattices from Tables VI and VII in Appendix C. Using the same notations as in Appendix D, one finds the following embeddings for these non-L-minimal CQHLs at $\sigma_H = 1$:

$$ {}_2(1)_2^4\,[3\,{}^{-1}3] \hookrightarrow \left\{ \begin{array}{c} {}_3(1)_2^6\,(2-1;0) \supset A_1 \\ {}_3(1)_2^8\,(1-1;1) \end{array} \right\} \hookrightarrow {}_5(1)_2^8\,(3 \mid {}^1A_1\,{}^1A_1\,{}^1A_1\,{}^1A_1) \hookrightarrow \ldots \tag{7.2} $$

We note that this chain of embeddings, with the corresponding possibilities of structural phase transitions, is particularly interesting in the light of the recent experimental data given in ref. 50. There evidence for a phase transition between different QH fluids at $\sigma_H = 1$ has been reported. The phase transition seems to be driven by an in-plane magnetic field $\mathbf{B}_c^{\parallel}$ and is observed in *double-layer* QH systems. Note that in (7.2), e.g., the first two

CQHLs [the lattice with symbol ${}_2(1)_2^4$ and the one with symbol ${}_3(1)_2^6$)] are both natural candidates for describing double-layer QH fluids. The first one can be interpreted as showing a discrete $\mathbb{Z}_2$ layer symmetry, while the second one can be thought to exhibit a continuous $A_1 = su(2)$ layer symmetry; see also the discussion in Section 1. Furthermore, since for all lattices in (7.2) the charge parameter λ equals 2, we would expect, as mentioned above, that quasiparticles with fractional charge 1/2 can be excited above the ground state of the corresponding QH fluids. An experimental investigation of this prediction would seem to be revealing and is encouraged!

APPENDIX A. SIMPLE LIE ALGEBRAS

The purpose of this appendix is to collect those facts about the simple Lie algebras $A_n = su(n-1)$, $n = 1, 2,\dots$, $D_n = so(2n)$, $n = 4, 5,\dots$, E_6, and E_7 which are basic for the classification of maximally symmetric CQHLs, as discussed in Section 5. For our explicit notations we adopt the conventions of ref. 45—they are followed, in particular, for the numbering of the simple roots of the algebras above, and we note that this numbering differs from the one chosen in ref. 46. Furthermore, for notational simplicity we often only write the symbol $\mathscr{G}$ denoting a simple Lie algebra when we are actually referring to the associated root lattice $\Gamma_{\mathscr{G}}$.

As stated in the text, the *ranks* of the Lie algebras A_n, D_n, and E_n and correspondingly of their associated root lattices are given by the index n in their symbols.

Further data about these algebras, which we generally denoted by $\mathscr{G}$, are given as follows: First, we specify the *Cartan matrices* $C(\mathscr{G})$ which characterize the associated root lattices $\Gamma_{\mathscr{G}}$ and we give the corresponding *discriminants* $\Delta(\mathscr{G}) = \det C(\mathscr{G})$. Second, we provide the *admissible weights* $\boldsymbol{\omega}$ in the dual lattices $\Gamma_{\mathscr{G}}^*$ by stating explicitly their dual-component vectors $\underset{\sim}{\omega}$, the so-called Dynkin labels. Moreover, the lengths squared, $\langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle$, and the orders, h_ω, of these weights in $\Gamma_{\mathscr{G}}^*/\Gamma_{\mathscr{G}}$ are listed.

• For $A_{m-1} = su(m)$, $m = 2, 3$, we have relative to a basis of simple roots $\{\mathbf{e}_1,\dots, \mathbf{e}_{m-1}\}$

$$C(A_{m-1}) = \left.\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix}\right\} m-1 \qquad \text{(A.1)}$$

with $\det C(A_{m-1}) = \mathbf{m}$.

The admissible weights $\boldsymbol{\omega}_t$, $t=1,\ldots,m-1$, correspond to the unitary irreducible representations (irreps) of $su(m)$ with "m-alities" t and dimensions $m\cdot(m-1)\cdots(m-t+1)/(1\cdot 2\cdots t)$. They are given by the dual-component vectors $\underset{\to}{\omega}_t=(\langle\boldsymbol{\omega}_t, \mathbf{e}_1\rangle,\ldots,\langle\boldsymbol{\omega}_t, \mathbf{e}_{m-1}\rangle)$ which read explicitly

$$\underset{\to}{\omega}_t=(\underbrace{0,\ldots,0,1,0,\ldots,0}_{m-1}),\qquad \text{with 1 in the } t\text{th position} \tag{A.2}$$

Moreover, their lengths squared and orders are given by

$$\langle\boldsymbol{\omega}_t, \boldsymbol{\omega}_t\rangle=\frac{t(m-t)}{m} \qquad \text{and} \qquad h_{\boldsymbol{\omega}_t}=\frac{m}{\gcd(m,t)} \tag{A.3}$$

We note that, from the point of view of characterizing CQHLs, the elementary weights $\boldsymbol{\omega}_t$ and $\boldsymbol{\omega}_{m-t}$ are equivalent; see the equivalence relation (3.1).

- For $D_n=so(2n)$, $n=4,5,\ldots$, we have

$$C(D_n)=\left.\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 0 & 2 \end{pmatrix}\right\} n \tag{A.4}$$

with $\det C(D_n)=4$.

There are three admissible weights, $\boldsymbol{\omega}_v$, $\boldsymbol{\omega}_s$, and $\boldsymbol{\omega}_{\bar{s}}$, corresponding to the $2n$-dimensional vector, the 2^{n-1}-dimensional spinor, and the conjugate spinor irrep of $so(2n)$, respectively. The corresponding n-dimensional dual-component vectors read

$$\begin{aligned} \underset{\to}{\omega}_v &= (1,0,\ldots,0) \\ \underset{\to}{\omega}_s &= (0,\ldots,0,1) \\ \underset{\to}{\omega}_{\bar{s}} &= (0,\ldots,0,1,0) \end{aligned} \tag{A.5}$$

Furthermore,

$$\langle\boldsymbol{\omega}_v, \boldsymbol{\omega}_v\rangle=1 \qquad \text{and} \qquad h_{\boldsymbol{\omega}_v}=2 \tag{A.6}$$

$$\langle\boldsymbol{\omega}_s, \boldsymbol{\omega}_s\rangle=\frac{n}{4}=\langle\boldsymbol{\omega}_{\bar{s}}, \boldsymbol{\omega}_{\bar{s}}\rangle \qquad \text{and} \qquad h_{\boldsymbol{\omega}_s}=h_{\boldsymbol{\omega}_{\bar{s}}}=\begin{cases}4\\2\end{cases} \quad \text{if } n \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases} \tag{A.7}$$

For the labeling of CQHLs, $\boldsymbol{\omega}_s$ and $\boldsymbol{\omega}_{\bar{s}}$ are equivalent by (3.1). Moreover, for D_4, all three admissible weights in (A.5) are equivalent [the so-called "triality" of $so(8)$].

- For E_6, we have

$$C(E_6)=\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \tag{A.8}$$

with det $C(E_6)=3$.

There are two admissible weights, $\boldsymbol{\omega}_f$ and $\boldsymbol{\omega}_{\bar{f}}$, corresponding to the 27-dimensional fundamental, and to its contragredient irrep of E_6, respectively. The corresponding dual-component vectors read

$$\begin{aligned} \underline{\omega}_f &= (1, 0, 0, 0, 0, 0) \\ \underline{\omega}_{\bar{f}} &= (0, 0, 0, 0, 1, 0) \end{aligned} \tag{A.9}$$

Furthermore,

$$\langle \boldsymbol{\omega}_f, \boldsymbol{\omega}_f \rangle = 4/3 = \langle \boldsymbol{\omega}_{\bar{f}}, \boldsymbol{\omega}_{\bar{f}} \rangle \qquad \text{and} \qquad h_{\boldsymbol{\omega}_f} = h_{\boldsymbol{\omega}_{\bar{f}}} = 3 \tag{A.10}$$

For the labeling of CQHLs, these two elementary weights are equivalent.

- Finally, for E_7 we have

$$C(E_7)=\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix} \tag{A.11}$$

with det $C(E_7)=2$.

There is one admissible weight, $\boldsymbol{\omega}_f$, corresponding to the 56-dimensional fundamental irrep of E_7, with

$$\underset{\rightarrow}{\omega}_f = (0, 0, 0, 0, 0, 1, 0) \tag{A.12}$$

and

$$\langle \boldsymbol{\omega}_f, \boldsymbol{\omega}_f \rangle = 3/2 \qquad \text{and} \qquad h_{\boldsymbol{\omega}_f} = 2 \tag{A.13}$$

APPENDIX B. MAXIMALLY SYMMETRIC CQHLS

In this appendix all maximally symmetric CQHLs with $l_{\text{min}} = l_{\text{max}} = L = 3$ and $\sigma_{\text{H}} < 1$ are listed. The compilation has been obtained by systematically exploiting Theorem 5.2 in Section 5 and the identities (5.9). The data are organized in Table V in 11 series (B1)–(B11), and for each series the following format is chosen:

First, the symbols of the CQHLs, ${}_N(n_{\text{H}}/d_{\text{H}})^g_\lambda$ are given; see (3.2). They are followed by the characterizing data of maximally symmetric CQHLs, $(L|{}^{\boldsymbol{\omega}}\Gamma_{\text{W}})$; see (5.4). Actually, since we are considering exclusively CQHLs with $L = 3$ in this appendix, the quantity L is omitted from the notation and only the data ${}^{\boldsymbol{\omega}}\Gamma_{\text{W}}$ is stated explicitly. If the Witt lattice is composite, $\Gamma_{\text{W}} = \Gamma_{\text{W}_1} \oplus \cdots \oplus \Gamma_{\text{W}_k}$, $k \geqslant 2$, and the elementary weight reads correspondingly $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \cdots + \boldsymbol{\omega}_k$, then we write ${}^{\boldsymbol{\omega}}\Gamma_{\text{W}} = {}^{\boldsymbol{\omega}_1}\Gamma_{\text{W}_1} \cdots {}^{\boldsymbol{\omega}_k}\Gamma_{\text{W}_k}$. As in Appendix A, the root lattices Γ_{W_i} are denoted by the symbols of the associated (simple) Lie algebras A_n, D_n, and E_n, respectively. Furthermore, the notation for the elementary weights $\boldsymbol{\omega}_t$, $\boldsymbol{\omega}_v$, $\boldsymbol{\omega}_s$, and $\boldsymbol{\omega}_f$ which are all given explicitly in Appendix A is simplified by only writing the indexing letters t, v, s and f, respectively. Finally, we adopt the convention of writing $a \mid b$ and $a \nmid b$ if a divides, respectively, does not divide b.

Second, for each series explicit examples of Hall fractions which can be realized by a CQHL of that series are given together with indications of their experimental status, typically in single-layer systems. For the corresponding notations see Fig. 1 in Section 1 and Table II in Section 5.

APPENDIX C. LOW-DIMENSIONAL, INDECOMPOSABLE CQHLS

The purpose of this appendix is to summarize the classification of all indecomposable CQHLs in two and three dimensions with relative-angular-momentum invariant $l_{\text{max}} \leqslant 5$ and of all such lattices in four dimensions with $l_{\text{max}} = 3$. We recall that, by definition [see (3.8)], we have $l_{\text{max}} = L_{\text{max}}$ for indecomposable CQHLs.

In Tables VI, VII, and IX, the CQHLs are organized according to increasing values of their Hall fractions σ_{H} and for each CQHL the symbol

Table V. All Maximally Symmetric CQHLs with $L=3$ and $\sigma_H<1$

	${}_N\left(\frac{n_H}{d_H}\right)^g_\lambda$	${}^{\omega}\Gamma_W$, parameters and examples
(B1)	${}_N\left(\frac{N}{2N+1}\right)^1_1$	${}^1A_{N-1}$, $N=1, 2,\ldots$: $\bullet\frac{1}{3}$ $\bullet\frac{2}{5}$ $\bullet\frac{3}{7}$ $\bullet\frac{4}{9}$ $\bullet\frac{5}{11}$ $\bullet\frac{6}{13}$ $\circ\frac{7}{15}$ $\circ\frac{8}{17}$ $\cdot\frac{9}{19}$ $\frac{10}{21}$ $\cdots$
(B2)	${}_N\left(\frac{1}{2}\right)^2_2$	${}^{\epsilon}D_{N-1}$, $N=3, 4,\ldots$: [Remark: ${}^{\epsilon}D_2\simeq{}^1A_1\,{}^1A_1$, ${}^{\epsilon}D_3\simeq{}^2A_3$, and ${}^{\epsilon}D_4\simeq{}^sD_4$] ${}^{(2)}\bullet\frac{1}{2}$
(B3)	${}_N\left(\frac{N}{N+4}\right)^g_\lambda$	${}^2A_{N-1}$, with $g=1$ (2) and $\lambda=1$ (1 or 2) if N is odd (even, and $4\nmid N$ or $4\mid N$); $N=5, 6,\ldots$: [Remark: ${}^2A_4\simeq$ "${}^{\surd}E_4$"] $\bullet\frac{5}{9}$ $\bullet\frac{3}{5}$ $\circ\frac{7}{11}$ $\bullet\frac{2}{3}$ $\circ\frac{9}{13}$ $\bullet\frac{5}{7}$ $\frac{11}{15}$ $\cdots$
(B4)	${}_{n_1+n_2-1}\left(\frac{n_1n_2/g\lambda}{(n_1n_2+n_1+n_2)/g\lambda}\right)^g_\lambda$	${}^1A_{n_1-1}\,{}^1A_{n_2-1}$, $n_1=gr_1$, $n_2=gr_2$ with $g=\gcd(n_1, n_2)$ and $\lambda=\gcd(r_1+r_2, g)$; $N=n_1+n_2-1=4, 5,\ldots$, and $2\leqslant n_1\leqslant n_2$: $n_1=2$: $\bullet\frac{6}{11}$ $\bullet\frac{4}{7}$ $\cdot\frac{10}{17}$ $\bullet\frac{3}{5}$ $\frac{14}{23}$ $\bullet\frac{8}{13}$ $\frac{18}{29}$ $\cdots$ $n_1=3$: $\bullet\frac{3}{5}$ $\frac{12}{19}$ $\frac{15}{23}$ $\bullet\frac{2}{3}$ $\frac{21}{31}$ $\cdots$ $\cdots$
(B5)	${}_N\left(\frac{n}{n+1}\right)^g_\lambda$	${}^1A_{n-1}\,{}^{\epsilon}D_{N-n}$, with $g=2$ (4) and $\lambda=2$ (1) if N is odd (even); $N=4, 5,\ldots$, and $2\leqslant n\leqslant N-2$: [For ${}^{\epsilon}D_2$, ${}^{\epsilon}D_3$, and ${}^{\epsilon}D_4$, see (B2)] $\bullet\frac{2}{3}$ $\frac{3}{4}$ $\bullet\frac{4}{5}$ $\frac{5}{6}$ ${}^{(2)}\circ\frac{6}{7}$ $\frac{7}{8}$ $\cdots$
(B6)	${}_N\left(\frac{N}{9}\right)^g_1$	${}^3A_{N-1}$, with $g=\gcd(N, 3)$; $N=6$, 7, and 8: $\bullet\frac{2}{3}$ $\frac{7}{9}$ $\frac{8}{9}$
(B7)	${}_N\left(\frac{4}{13-N}\right)^g_1$	${}^sD_{N-1}$, with $g=2$ (1) if N is odd (even); $N=6$, 7, and 8: [Remark: ${}^sD_5\simeq$ "${}^{\surd}E_5$"] $\bullet\frac{4}{7}$ $\bullet\frac{2}{3}$ $\bullet\frac{4}{5}$
(B8)	$\bullet{}_7\left(\frac{3}{5}\right)^1_1$	${}^{\surd}E_6$
	$\bullet{}_8\left(\frac{2}{3}\right)^1_1$	${}^{\surd}E_7$

Table V. Continued

	${}_N\left(\frac{n_H}{d_H}\right)^g_\lambda$	${}^\omega\Gamma_W$, parameters and examples
(B9)	${}_N\left(\frac{2N-2}{N+7}\right)^g_1$	${}^1A_1\,{}^2A_{N-2}$, with $g=2\,(1)$ if N is odd (even); $N=6$, 7, and 8: $\frac{10}{13}$ $(2)^\circ\frac{6}{7}$ $\frac{14}{15}$
(B10)	$\bullet\,{}_7\left(\frac{4}{5}\right)^2_1$	${}^1A_1\,{}^sD_5$ [Remark: ${}^sD_5 \simeq$ "${}^fE_5$"]
	$(2)^\circ\,{}_8\left(\frac{6}{7}\right)^1_1$	${}^1A_1\,{}^fE_6$
	${}_7\left(\frac{15}{17}\right)^1_1$	${}^1A_2\,{}^2A_4$ [Remark: ${}^2A_4 \simeq$ "${}^fE_4$"]
	${}_8\left(\frac{12}{13}\right)^1_1$	${}^1A_2\,{}^sD_5$
	${}_8\left(\frac{20}{21}\right)^1_1$	${}^1A_3\,{}^2A_4$
(B11)	${}_N\left(\frac{6N-18}{5N-9}\right)^g_1$	${}^1A_1\,{}^1A_2\,{}^1A_{N-4}$, with $g=3$, 2, 1 for $N=6$, 7, and 8: $(2)^\circ\frac{6}{7}$ $\frac{12}{13}$ $\frac{30}{31}$

${}_N(n_H/d_N)^g_\lambda$ is given together with indications of the experimental status of the corresponding Hall fraction. For the latter indications, notations are as in Appendix B. The symbols are followed by the explicit data $(K, \underset{\to}{Q})$ which characterize the CQHLs completely; see the beginning of Section 3. For a succinct presentation of the data $(K, \underset{\to}{Q})$, we choose symmetric bases in the corresponding CQHLs [see (3.5)] and adopt the following notations:

$$N=2: \quad [l_{\min}\,{}^a\,l_{\max}] \quad \text{for} \quad K=\begin{pmatrix} l_{\min} & a \\ a & l_{\max}\end{pmatrix} \quad \underset{\to}{Q}=(1,1) \tag{C.1}$$

$$N=3: \quad (\underset{\bullet}{a_1}a_2;b) \quad \text{for} \quad K=\begin{pmatrix} 3 & a_1 & a_2 \\ a_1 & 3 & b \\ a_2 & b & 3\end{pmatrix}, \quad \underset{\to}{Q}=(1,1,1) \tag{C.2}$$

$$N=4: \quad (a_1a_2a_3;b_1b_2;c) \quad \text{for} \quad K=\begin{pmatrix} 3 & a_1 & a_2 & a_3 \\ a_1 & 3 & b_1 & b_3 \\ a_2 & b_1 & 3 & c \\ a_3 & b_2 & c & 3\end{pmatrix}, \quad \underset{\to}{Q}=(1,1,1,1) \tag{C.3}$$

Table VI. All Indecomposable CQHLs with $N=2$ and $3 \leqslant l_{\min} \leqslant l_{\max} \leqslant 5$

	${}_N\left(\frac{n_H}{d_H}\right)_\lambda^g$	$[l_{\min}\,{}^a l_{\max}]$	Remarks
$0<\sigma_H<\frac{1}{5}$	None, by (4.3)		
$\Sigma_2^+,\ \frac{1}{5}\leqslant\sigma_H<\frac{1}{4}$	$\bullet\,{}_2(\frac{2}{9})_1^1$	$[5^4 5]$	$=(5\mid {}^1A_1)=\mathscr{S}_2({}_1(1)_1^1\oplus{}_1(1)_1^1)$
$\Sigma_2^-,\ \frac{1}{4}\leqslant\sigma_H<\frac{1}{3}$	${}_2(\frac{1}{4})_2^2$	$[5^3 5]$	$=\mathscr{S}_1({}_2(\frac{1}{2})_2^2)$
	$\circ\,{}_2(\frac{2}{7})_1^3$	$[5^2 5]$	$=\mathscr{S}_1({}_1(\frac{1}{3})_1^1\oplus{}_1(\frac{1}{3})_1^1)$
$\Sigma_1^+,\ \frac{1}{3}\leqslant\sigma_H<\frac{1}{2}$	$\bullet\,{}_2(\frac{1}{3})_2^4$	$[5^1 5]$	$=\mathscr{S}_1({}_2(1)_2^4)$
	$\cdot\,{}_2(\frac{4}{11})_1^1$	$[3^2 5]$	$=\mathscr{S}_1({}_1(1)_1^1\oplus{}_1(\frac{1}{3})_1^1)$
	$\bullet\,{}_2(\frac{2}{5})_1^1$	$[3^2 3]$	$=(3\mid {}^1A_1)=\mathscr{S}_1({}_1(1)_1^1\oplus{}_1(1)_1^1)$
	$\bullet\,{}_2(\frac{3}{7})_1^2$	$[3^1 5]$	
$\Sigma_1^-,\ \frac{1}{2}\leqslant\sigma_H<1$	${}_{(2)}\bullet\,{}_2(\frac{1}{2})_2^2$	$[3^1 3]$	
	${}_{(2)}\bullet\,{}_2(\frac{1}{2})_2^6$	$[5^{-1} 5]$	
	${}_{B,n\text{-}p}\bullet\,{}_2(\frac{2}{3})_1^7$	$[5^{-2} 5]$	
	${}_{(B\text{-}p)}\bullet\,{}_2(\frac{5}{7})_1^2$	$[3^{-1} 5]$	
$\Sigma_0^+,\ 1\leqslant\sigma_H<\infty$	$\bullet\,{}_2(1)_2^4$	$[3^{-1} 3]$	
	$\bullet\,{}_2(1)_2^8$	$[5^{-3} 5]$	
	${}_2(\frac{12}{11})_1^1$	$[3^{-2} 5]$	
	$\bullet\,{}_2(2)_1^5$	$[3^{-2} 3]$	$\supset A_1$
	$\bullet\,{}_2(2)_1^9$	$[5^{-4} 5]$	$\supset A_1$

Furthermore, in Tables VI–IX we indicate as remarks the corresponding Witt sublattices and/or preimages under the shift maps when they exist. We note that in Tables VI and VII, the Witt sublattices of the CQHLs with $\sigma_H \geqslant 2$ are not fully included in their neutral sublattices, i.e., some of the associated symmetry generators have a non-vanishing electric charge.

In Table VIII the symbols of a physically relevant subset of all three-dimensional, indecomposable CQHLs with $l_{\max}=5$ are provided. They are organized according to the values of their relative-angular-momentum invariants $[l_{\min}, l_2, l_{\max}]$; see (6.1). The symbols are followed by triples $(a_1 a_2; b)$ which have the same meaning as in (C.2) above with the only

Table VII. All Indecomposable CQHLs with $N=3$ and $l_{min}=l_{max}=3$

	${}_N\left(\frac{n_H}{d_H}\right)_\lambda^g$	$(a_1 a_2; b)$	Remarks
$0<\sigma_H<\frac{1}{3}$	None, by (4.3)		
$\Sigma_1^+,\ \frac{1}{3}\leqslant\sigma_H<\frac{1}{2}$	$\bullet_3\left(\frac{3}{7}\right)_1^1$	$(2\ 2;\ 2)$	$=(3\mid {}^1A_2)=\mathscr{S}_1({}_1(1)_1^1\oplus{}_1(1)_1^1\oplus{}_1(1)_1^1)$
$\Sigma_1^-,\ \frac{1}{2}\leqslant\sigma_H<1$	${}_{(2)}\bullet_3\left(\frac{1}{2}\right)_2^2$	$(2\ 1;\ 2)$	$=(3\mid {}^1A_1\ {}^1A_1)$
	$\bullet_3\left(\frac{7}{13}\right)_1^1$	$(2\ 1;\ 1)$	$\supset A_1$
	${}_{B\text{-}p}\bullet_3\left(\frac{3}{5}\right)_1^4$	$(1\ 1;\ 1)$	
	${}_{B,n\text{-}p}\bullet_3\left(\frac{2}{3}\right)_1^4$	$(2\ 0;\ 1)$	$\supset A_1$
	${}_{(B\text{-}p)}\bullet_3\left(\frac{5}{7}\right)_1^3$	$(1\ 0;\ 1)$	
$\Sigma_0^+,\ 1\leqslant\sigma_H<2$	$\bullet_3(1)_2^6$	$(2\ -1;\ 0)$	$\supset A_1$
	$\bullet_3(1)_2^8$	$(1\ -1;\ 1)$	
	${}_3\left(\frac{23}{21}\right)_1^1$	$(1\ -1;\ 0)$	
	${}_3\left(\frac{15}{13}\right)_1^1$	$(2\ -1;\ -1)$	$\supset A_1$
	$\bullet_3\left(\frac{7}{5}\right)_1^4$	$(1\ -1;\ -1)$	
	${}_3\left(\frac{13}{7}\right)_1^3$	$(0\ -1;\ -1)$	
$2\leqslant\sigma_H<3$	$\bullet_3(2)_1^8$	$(2\ -2;\ -1)$	$\supset A_1\ A_1$
	$\bullet_3(2)_1^{12}$	$(1\ -2;\ 0)$	$\supset A_1$
	${}_3\left(\frac{31}{13}\right)_1^1$	$(1\ -2;\ -1)$	$\supset A_1$
	$\circ_3\left(\frac{19}{7}\right)_1^1$	$(2\ -2;\ -2)$	$\supset A_2$
$3\leqslant\sigma_H<\infty$	$\bullet_3(3)_1^{16}$	$(-1\ -1;\ -1)$	
	${}_3\left(\frac{11}{3}\right)_2^2$	$(0\ -2;\ -1)$	$\supset A_1$
	${}_3\left(\frac{9}{2}\right)_2^2$	$(1\ -2;\ -2)$	$\supset A_1\ A_1$

change that the diagonal elements of K are not 3 3 3, but are given, from left to right, by $l_{\min} l_2 l_{\max}$, as specified at the beginning of each sublist. Moreover, in the sublist with invariants [5, 5, 5], all those inverse images under the shift map $\mathscr{S}_1$ are indicated which belong to Table VII with invariants [3, 3, 3]; see (4.12). Since the invariants N, g, and λ do not change under the shift maps, they are suppressed in the labeling of the inverse images. Finally, only CQHLs with $\lambda d_H\leqslant 22$ are listed. For the

Table VIII. Symbols of Indecomposable CQHLs with $N=3$, $3 \leqslant l_{\min} \leqslant l_{\max} = 5$, and $\sigma_H < 1$[a]

$[l_{\min}, l_2, l_{\max}] = [3, 3, 5]$:

${}_3(\frac{7}{17})_1^1\,(2\;2;\,2)$	${}_3(\frac{4}{9})_1^2\,(2\;1;\,2)$	${}_3(\frac{1}{2})_2^6\,(1\;2;\,2)$	${}_3(\frac{6}{11})_1^2\,(2\;0;\,1)$
${}_3(\frac{5}{9})_1^4\,(1\;1;\,1)$	${}_3(\frac{4}{7})_1^4\,(1\;0;\,2)$	${}_3(\frac{2}{3})_1^{10}\,(0\;1;\,2)$	${}_3(\frac{9}{13})_1^3\,(0\;1;\,1)$
${}_3(\frac{8}{11})_1^2\,(2\;0;\,-1)$	${}_3(\frac{3}{4})_2^4\,(1\;1;\,-1)$	${}_3(\frac{4}{5})_1^6\,(0\;2;\,-1)$	...

(in total 17 CQHLs)

$[l_{\min}, l_2, l_{\max}] = [3, 5, 5]$:

${}_3(\frac{7}{19})_1^1\,(3\;2;\,3)$	${}_3(\frac{3}{8})_2^2\,(2\;2;\,3)$	${}_3(\frac{5}{13})_1^3\,(2\;2;\,2)$	${}_3(\frac{2}{5})_2^4\,(2\;2;\,1)$
${}_3(\frac{3}{7})_1^5\,(2\;1;\,3)$	${}_3(\frac{5}{11})_1^4\,(1\;1;\,3)$	${}_3(\frac{9}{19})_1^3\,(1\;1;\,2)$	${}_3(\frac{1}{2})_4^8\,(1\;1;\,1)$
${}_3(\frac{7}{13})_1^5\,(1\;1;\,0)$	${}_3(\frac{3}{5})_1^{12}\,(1\;;\,-1)$	${}_3(\frac{8}{13})_1^4\,(2\;0;\,-1)$	${}_3(\frac{5}{7})_1^7\,(1\;1;\,-2)$
${}_3(\frac{11}{15})_1^4\,(1\;-1;\,1)$	${}_3(\frac{13}{17})_1^3\,(2\;-1;\,-1)$	${}_3(\frac{9}{11})_1^4\,(-1\;-1;\,3)$	${}_3(\frac{7}{8})_2^4\,(1\;-1;\,-1)$
${}_3(\frac{17}{19})_1^3\,(-1\;-1;\,2)$	...		

(in total 34 CQHLs)

$[l_{\min}, l_2, l_{\max}] = [5, 5, 5]$:

${}_3(\frac{3}{13})_1^1 = \mathscr{S}_1(\frac{3}{7})$	${}_3(\frac{1}{4})_2^2 = \mathscr{S}_1(\frac{1}{2})$	${}_3(\frac{3}{11})_1^4 = \mathscr{S}_1(\frac{3}{5})$	${}_3(\frac{2}{7})_1^4 = \mathscr{S}_1(\frac{2}{3})$
${}_3(\frac{5}{17})_1^3 = \mathscr{S}_1(\frac{5}{7})$	${}_3(\frac{1}{3})_2^6 = \mathscr{S}_1(1)$	${}_3(\frac{1}{3})_2^8 = \mathscr{S}_1(1)$	${}_3(\frac{1}{3})_3^9\,(2\;2;\,2)$
${}_3(\frac{4}{11})_2^4\,(2\;2;\,1)$	${}_3(\frac{7}{19})_1^4 = \mathscr{S}_1(\frac{7}{5})$	${}_3(\frac{2}{5})_1^8 = \mathscr{S}_1(2)$	${}_3(\frac{2}{5})_1^{12} = \mathscr{S}_1(2)$
${}_3(\frac{7}{17})_1^5\,(2\;0;\,2)$	${}_3(\frac{3}{7})_1^{16} = \mathscr{S}_1(3)$	${}_3(\frac{1}{2})_2^{10}\,(4\;0;\,-1)$	${}_3(\frac{1}{2})_2^{16}\,(3\;1;\,-1)$
${}_3(\frac{1}{2})_2^{18}\,(2\;2;\,-1)$	${}_3(\frac{5}{9})_1^{12}\,(1\;1;\,-1)$	${}_3(\frac{11}{19})_1^4\,(3\;-1;\,-1)$	${}_3(\frac{7}{11})_1^9\,(2\;-1;\,-1)$
${}_3(\frac{2}{3})_1^{12}\,(4\;-2;\,-1)$	${}_3(\frac{2}{3})_1^{20}\,(3\;0;\,-2)$	${}_3(\frac{2}{3})_1^{24}\,(2\;;\,-2)$	${}_3(\frac{9}{13})_1^7\,(1\;1;\,-2)$
${}_3(\frac{5}{7})_2^8\,(1\;-1;\,-1)$	...		

(in total 48 CQHLs)

[a] Dots (...) indicate omitted fractions with $\lambda d_H > 22$.

physical interpretation of λd_H as the smallest possible (fractional) charge of quasiparticle excitations in the corresponding QH fluids see (3.4).

APPENDIX D. EMBEDDINGS OF L-MINIMAL CQHLS

In this appendix, embeddings [see (7.1)] of L-minimal CQHLs with Hall fractions in the window $\sigma_H \in \Sigma_1^- = [1/2, 1)$ are listed. More precisely,

Table IX. All Indecomposable CQHLs with $N = 4$, $l_{min} = l_{max} = 3$, and $\sigma_H < 1$

	${}_N\left(\frac{n_H}{d_H}\right)^g_\lambda$	$(a_1 a_2 a_3; b_1 b_2; c)$	Remarks
$0 < \sigma_H < \frac{1}{3}$	None, by (4.3)		
Σ_1^+, $\frac{1}{3} \leqslant \sigma_H < \frac{1}{2}$	$\bullet_4\left(\frac{4}{9}\right)^1_1$	$(2\,2\,2;\,2\,2;\,2)$	$=(3 \mid {}^1A_3)$
Σ_1^-, $\frac{1}{2} \leqslant \sigma_H < \frac{2}{3}$	${}_{(2)}\bullet_4\left(\frac{1}{2}\right)^2_2$	$(2\,2\,1;\,2\,2;\,2)$	$=(3 \mid {}^2A_3)$
	$\bullet_4\left(\frac{6}{11}\right)^1_1$	$(2\,2\,1;\,2\,1;\,2)$	$=(3 \mid {}^1A_1\,{}^1A_2)$
	$\bullet_4\left(\frac{5}{9}\right)^2_1$	$(2\,2\,1;\,2\,1;\,1)$	$\supset A_2$
	$\bullet_4\left(\frac{4}{7}\right)^3_1$	$(2\,1\,1;\,1\,1;\,2)$	$\supset A_1 A_1$
	${}_{B\text{-}p}\bullet_4\left(\frac{3}{5}\right)^4_1$	$(2\,1\,1;\,2\,1;\,1)$	$\supset A_1\,A_1$
	$\bullet_4\left(\frac{5}{8}\right)^2_2$	$(2\,1\,1;\,1\,1;\,1)$	$\supset A_1$
$\frac{2}{3} \leqslant \sigma_H < 1$	${}_{B,n\text{-}p}\bullet_4\left(\frac{2}{3}\right)^4_1$	$(2\,1\,0;\,2\,1;\,2)$	$=(3 \mid {}^1A_1\,{}^1A_1\,{}^1A_1)$
	${}_{B,n\text{-}p}\bullet_4\left(\frac{2}{3}\right)^5_1$	$(2\,2\,0;\,2\,1;\,1)$	$\supset A_2$
	${}_{B,n\text{-}p}\bullet_4\left(\frac{2}{3}\right)^8_2$	$(1\,1\,1;\,1\,1;\,1)$	
	${}_{(B\text{-}p)}\bullet_4\left(\frac{5}{7}\right)^4_1$	$(2\,1\,0;\,1\,1;\,1)$	$\supset A_1$
	$\bullet_4\left(\frac{8}{11}\right)^3_1$	$(2\,1\,1;\,1\,1;\,0)$	$\supset A_1$
	${}_4\left(\frac{14}{19}\right)^1_1$	$(2\,1\,0;\,2\,0;\,1)$	$\supset A_1\,A_1$
	${}_4\left(\frac{3}{4}\right)^2_2$	$(2\,2\,0;\,2\,0;\,1)$	$\supset A_2$
	${}_4\left(\frac{3}{4}\right)^6_2$	$(1\,1\,0;\,1\,1;\,1)$	
	${}_4\left(\frac{13}{17}\right)^2_1$	$(2\,1\,0;\,1\,0;\,1)$	$\supset A_1$
	$\bullet_4\left(\frac{4}{5}\right)^9_1$	$(1\,1\,0;\,0\,1;\,1)$	
	${}_4\left(\frac{26}{31}\right)^1_1$	$(2\,0\,0;\,1\,0;\,1)$	$\supset A_1$
	${}_4\left(\frac{11}{13}\right)^4_1$	$(1\,1\,0;\,1\,0;\,1)$	
	${}_{(2)}\circ_4\left(\frac{6}{7}\right)^4_1$	$(2\,0\,0;\,1\,1;\,1)$	$\supset A_1$
	${}_4\left(\frac{10}{11}\right)^5_1$	$(1\,0\,0;\,1\,0;\,1)$	

in accordance with the results presented in Section 7, we take the following sets of CQHLs into account: (i) all generic, L-minimal CQHLs in low dimensions, $N \leqslant 4$ (see Appendix C), (ii) all maximally symmetric, L-minimal CQHLs in dimensions $N \leqslant 10$ (see Appendix B), and (iii) all composites of two identical lattices belonging to the prominent A-series given by (B1) in Appendix B. In Table X, CQHLs are specified by their symbols,

Table X. All Embeddings of L-Minimal CQHLs That Have $\sigma_H \in \Sigma_1^-$ and Belong to the Classes Mentioned Above

---

${}_{(2)}\bullet\frac{1}{2}$

$${}_2(\tfrac12)^2_2\,[3^13] \hookrightarrow {}_3(\tfrac12)^2_2\,{}^1A_1\,{}^1A_1 \hookrightarrow {}_4(\tfrac12)^2_2\,{}^2A_3 \hookrightarrow {}_5(\tfrac12)^2_2\,{}^rD_4 \hookrightarrow \cdots$$

$\bullet\frac{5}{9}$

$${}_4(\tfrac59)^2_1\,(2\,2\,1;2\,1;1) \supset A_2 \hookrightarrow {}_5(\tfrac59)^1_1\,{}^2A_4$$

$\bullet\frac{4}{7}$

$${}_4(\tfrac47)^3_1\,(2\,1\,1;1\,1;2) \supset A_1\,A_1 \hookrightarrow {}_5(\tfrac47)^2_1\,{}^1A_1\,{}^1A_3 \hookrightarrow {}_6(\tfrac47)^1_1\,{}^sD_5$$

${}_{B\text{-}p}\bullet\frac{3}{5}$

$${}_3(\tfrac35)^4_1\,(1\,1;1) \hookrightarrow {}_4(\tfrac35)^4_1\,(2\,1\,1;2\,1;1) \supset A_1\,A_1 \hookrightarrow$$
$$\hookrightarrow {}_5(\tfrac35)^3_1\,{}^1A_2\,{}^1A_2 \hookrightarrow \left.\begin{matrix} {}_6(\tfrac35)^2_1\,{}^2A_5 \\ {}_7(\tfrac35)^2_2\,{}^1A_1\,{}^1A_5 \end{matrix}\right\} \hookrightarrow {}_7(\tfrac35)^1_1\,{}^fE_6$$

${}_{B,n\text{-}p}\bullet\frac{2}{3}$

$${}_1(\tfrac13)^1_1\,[3] \oplus {}_1(\tfrac13)^1_1\,[3] \hookrightarrow {}_3(\tfrac23)^4_1\,(2\,0;1\,2) \supset A_1 \hookrightarrow$$
$$\hookrightarrow \left\{\begin{matrix} {}_4(\tfrac23)^5_1\,(2\,2\,0;2\,1;1) \supset A_2 \\ {}_4(\tfrac23)^4_1\,{}^1A_1\,{}^1A_1\,{}^1A_1 \end{matrix}\right\} \hookrightarrow {}_5(\tfrac23)^4_1\,{}^2A_3\,{}^1A_1 \hookrightarrow$$
$$\hookrightarrow \left.\begin{matrix} {}_6(\tfrac23)^4_1\,{}^1A_1\,{}^rD_4 \\ {}_6(\tfrac23)^3_1\,{}^3A_5 \\ {}_7(\tfrac23)^4_2\,{}^1A_3\,{}^1A_3 \end{matrix}\right\} \hookrightarrow \left.\begin{matrix} {}_7(\tfrac23)^4_1\,{}^1A_1\,{}^rD_5 \hookrightarrow {}_8(\tfrac23)^4_1\,{}^1A_1\,{}^rD_6 \hookrightarrow \cdots \\ \hookrightarrow {}_7(\tfrac23)^2_1\,{}^sD_6 \\ {}_8(\tfrac23)^2_2\,{}^2A_7 \\ {}_8(\tfrac23)^3_1\,{}^1A_2\,{}^1A_5 \end{matrix}\right\} \hookrightarrow {}_8(\tfrac23)^1_1\,{}^fE_7$$

${}_{(B\text{-}p)}\bullet\frac{5}{7}$

$${}_3(\tfrac57)^3_1\,(1\,0;1) \hookrightarrow {}_4(\tfrac57)^4_1\,(2\,1\,0;1\,1;1) \supset A_1$$

$\frac{3}{4}$

$$\left.\begin{matrix} {}_4(\tfrac34)^2_2\,(2\,2\,0;2\,0;1) \supset A_2 \\ {}_4(\tfrac34)^6_2\,(1\,1\,0;1\,1;1) \end{matrix}\right\} \hookrightarrow {}_5(\tfrac34)^2_2\,{}^1A_2\,{}^1A_1\,{}^1A_1 \hookrightarrow {}_6(\tfrac34)^2_2\,{}^1A_2\,{}^2A_3 \hookrightarrow \cdots$$

$\bullet\frac{4}{5}$

$$\left.\begin{matrix} {}_2(\tfrac25)^1_1\,{}^1A_1 \oplus {}_2(\tfrac25)^1_1\,{}^1A_1 \\ {}_4(\tfrac45)^9_1\,(1\,1\,0;0\,1;1) \end{matrix}\right\} \hookrightarrow {}_6(\tfrac45)^4_1\,{}^1A_3\,{}^1A_1\,{}^1A_1 \hookrightarrow$$
$$\hookrightarrow \left\{\begin{matrix} {}_7(\tfrac45)^4_1\,{}^1A_3\,{}^2A_3 \hookrightarrow \cdots \\ {}_7(\tfrac45)^2_1\,{}^1A_1\,{}^sD_5 \hookrightarrow {}_7(\tfrac45)^1_1\,{}^sD_7 \end{matrix}\right.$$

$\frac{5}{6}$

$${}_7(\tfrac56)^2_2\,{}^1A_4\,{}^1A_1\,{}^1A_1 \hookrightarrow {}_8(\tfrac56)^2_2\,{}^1A_4\,{}^2A_3 \hookrightarrow \cdots$$

${}_{(2)}\circ\frac{6}{7}$

$${}_4(\tfrac67)^4_1\,(2\,0\,0;1\,1;1) \supset A_1 \hookrightarrow \left.\begin{matrix} {}_6(\tfrac67)^3_1\,{}^1A_1\,{}^1A_2\,{}^1A_2 \\ {}_3(\tfrac37)^1_1\,{}^1A_2 \oplus {}_3(\tfrac37)^1_1\,{}^1A_2 \end{matrix}\right\} \hookrightarrow$$
$$\left.\begin{matrix} \hookrightarrow {}_7(\tfrac67)^2_1\,{}^1A_1\,{}^2A_5 \\ {}_8(\tfrac67)^4_1\,{}^1A_5\,{}^1A_1\,{}^1A_1 \end{matrix}\right\} \begin{matrix} \hookrightarrow {}_8(\tfrac67)^1_1\,{}^1A_1\,{}^fE_6 \\ \hookrightarrow {}_9(\tfrac67)^4_1\,{}^1A_5\,{}^2A_3 \hookrightarrow \cdots \end{matrix}$$

$\frac{7}{8}$

$${}_9(\tfrac78)^2_2\,{}^1A_6\,{}^1A_1\,{}^1A_1 \hookrightarrow {}_{10}(\tfrac78)^2_2\,{}^1A_2\,{}^2A_3 \hookrightarrow \cdots$$

${}_N(n_H/d_H)^g_\lambda$, and the explicit data characterizing their structure. These data are given in the conventions chosen in Appendices B and C, respectively.

In order to simplify notation in the subsequent table, we note that, at the fractions $\sigma_H = n/(n+1)$, $n = 1, 2, \ldots$, there are infinite "chains" of embeddings,

$$
\begin{aligned}
&{}_{n+2}\left(\frac{n}{n+1}\right)^g_\lambda {}^1A_{n-1}\,{}^1A_1\,{}^1A_1 \hookrightarrow {}_{n+3}\left(\frac{n}{n+1}\right)^g_\lambda {}^1A_{n-1}\,{}^2A_3 \\
&\hookrightarrow {}_{n+4}\left(\frac{n}{n+1}\right)^g_\lambda {}^1A_{n-1}\,{}^vD_4 \hookrightarrow \cdots \hookrightarrow {}_N\left(\frac{n}{n+1}\right)^g_\lambda {}^1A_{n-1}\,{}^vD_{N-n} \hookrightarrow \cdots
\end{aligned}
\tag{D.1}
$$

In Table X the respective next members of these chains of embeddings are understood when we write the dots ... .

APPENDIX E. HIERARCHY QH LATTICES

In this appendix we collect some basic facts about the description of the Haldane–Halperin[29] and the Jain–Goldman[30] hierarchy fluids in terms of QH lattices (Γ, $\mathbf{Q}$). First, we follow the ideas presented by Read.[9]

The Gram matrix K [see (2.2)] which characterizes the integral lattice Γ associated with a hierarchy fluid with Hall conductivity $\sigma_H = n_H/d_H$, where d_H is *odd*, can be read off from the "continued fraction expansion" of σ_H. Let

$$
\sigma_H = \frac{n_H}{d_H} = \cfrac{1}{m - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_{N-1}}}}}}
\tag{E.1}
$$

where m is an *odd*, positive integer, and $a_1, \ldots, a_{N-1}$ are *even* integers of either sign. Then the associated Gram matrix K is given by

$$
(K_{ij}) = \left.\begin{pmatrix}
m & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
-1 & a_1 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & -1 & a_2 & -1 & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & a_{N-1}
\end{pmatrix}\right\} N
\tag{E.2}
$$

which we abbreviate by the symbol

$$[m; a_1, \ldots, a_{N-1}] \tag{E.3}$$

We note that the signs of the 1's in (E.2) can be changed by suitable equivalence transformations (3.1). The choice of all the negative signs in (E.2) is our convention. Moreover, we remark that, *from a QH lattice point of view, the two hierarchy schemes of Haldane–Halperin*[29] *and Jain–Goldman*[30] *are equivalent*; see (3.1) and also the examples below. For this reason, we simply talk about "hierarchy QH lattices."

In the dual basis associated with (E.2), the integer-valued linear functional (or charge vector) **Q** is given by

$$\underset{\rightarrow}{Q} = (\underbrace{1, 0, \ldots, 0}_{N}) \tag{E.4}$$

See the beginning of Section 3.

With the help of Kramer's rule (2.5), one easily verifies that

$$\sigma_{\mathrm{H}} = \langle \mathbf{Q}, \mathbf{Q} \rangle = \underset{\rightarrow}{Q} \cdot K^{-1} \underset{\rightarrow}{Q}^{T} \tag{E.5}$$

From Eqs. (E.2) and (E.4) it is clear that the charge vector **Q** is *primitive* and *odd*, as defined in (2.6) and (2.7), respectively.

We note that in general the integral lattice Γ specified by (E.2) is *not* Euclidean. In order for it to be *Euclidean*, the Gram matrix K in (E.2) has to be positive-definite. One can show that K is positive-definite if and only if all the coefficients a_i, $i = 1, \ldots, N-1$, are *positive*. In this situation, the hierarchy QH lattice $(\Gamma, \mathbf{Q})$ is a CQHL, as defined in Section 2. In particular, it satisfies assumption (A5) there.

In the remaining part of this appendix we comment on the status of assumption (A5) for the *non-Euclidean* hierarchy fluids. All (Euclidean *and* non-Euclidean) hierarchy QH lattices satisfy assumptions (A1)–(A4) of Section 2.

We exemplify the situation of non-Euclidean hierarchy QH lattices by discussing in some detail the two physically important series of hierarchy fluids with $\sigma_{\mathrm{H}} = N/(2N-1)$, and $N/(4N-1)$, $N \geqslant 2$.

(a) $\sigma_{\mathrm{H}} = N/(2N-1)$: By (E.2) and (E.3) the Gram matrices K of these hierarchy fluids are given by

$$K = [1; \underbrace{-2, \ldots, -2}_{N-1}] \tag{E.6}$$

and the charge vectors **Q** are given by (E.4). In order to make the lattice structures behind (E.6) more explicit, we apply equivalence transformations (3.1), with S given by

$$S = \left.\begin{pmatrix} 1 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & -1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \pm 1 \end{pmatrix}\right\} N \tag{E.7}$$

We find

$$K' = [1] \oplus (-1) \cdot [3; \underbrace{2, \ldots, 2}_{N-2}]$$

and

$$\underset{\to}{Q} = 1 + (\underbrace{-1, 0, \ldots, 0}_{N-1}) \tag{E.8}$$

The interpretation of (E.8) is that, from a QH lattice point of view, the hierarchy fluids at $\sigma_{\mathrm{H}} = N/(2N-1) = 1 - (N-1)/[2(N-1)+1]$ are indeed the "charge conjugates" of the "elementary" $(N-1)/[2(N-1)+1]$ fluids exhibiting $su(N-1)$-current algebras at level 1; see example (c) at the end of Section 3.

We note that from (E.8) it is clear that these non-Euclidean hierarchy QH lattices satisfy assumption (A5) of Section 2.

(b) $\sigma_{\mathrm{H}} = N/(4N-1)$: By (E.2) and (E.3) the Gram matrices K of these hierarchy fluids read

$$K = [3; \underbrace{-2, \ldots, -2}_{N-1}] \tag{E.9}$$

and the charge vectors **Q** are given by (E.4). Again, in order to make the composite nature of the lattices described by (E.6) explicit, we apply equivalence transformations (3.1), with S given by

$$S=\left.\begin{pmatrix} 2N-1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ 2N-2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ -(2N-4) & 0 & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ 2N-6 & 0 & 0 & -1 & \cdot & \cdot & \cdot & -1 \\ -(2N-8) & 0 & \cdot & 0 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \pm 2 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \mp 1 \end{pmatrix}\right\} N \tag{E.10}$$

This results in

$$\begin{aligned} K' &= [4N-1] \oplus (-1)\cdot(\underbrace{[1] \oplus \cdots \oplus [1]}_{N-1}) \\ \underset{=}{Q} &= (2N-1) \oplus \underbrace{(-1) \oplus \cdots \oplus (-1)}_{N-1} \end{aligned} \tag{E.11}$$

At the level of Hall conductivities the decompositions (E.11) can be expressed as

$$\sigma_{\mathrm{H}} = N/(4N-1) = (2N-1)^2/(4N-1) - 1 - \cdots - 1$$

with $N-1$ summands of -1.

Hence, similarly to (a), the lattices Γ of this series are composed of positive- *and* negative-definite sublattices Γ_e and Γ_h, respectively. Contrary to (a), however, it follows from (E.11) that the restrictions of the charge vector $\mathbf{Q}$ to the positive- and negative-definite components of Γ, $\mathbf{Q}_e$ and $\mathbf{Q}_h$, respectively [see (2.8) and (2.9)], are *not separately* primitive. Rather, it is only the *full* integer-valued linear form $\mathbf{Q} = \mathbf{Q}_e \oplus \mathbf{Q}_h \in \Gamma^* = \Gamma_e^* \oplus \Gamma_h^*$ which is primitive; see (2.6).

In physical terms this means that, similar to assumption (A5) in Section 2, the dynamics of the positively and of the negatively charged (quasi)particle-rich subfluids—corresponding to Γ_e and Γ_h, respectively—are independent in the scaling limit. Contrary to (A5), however, the physics of these two subfluids are *not* identical up to charge conjugation. [The pair $(\Gamma_e, \mathbf{Q}_e)$ is *not* a CQHL, as defined in Section 2, since $\mathbf{Q}_e$ is *not* primitive]. We note that the fundamental charge carriers of these QH fluids, electrons *and* holes, are described as *composites* of the "basic" positively and negatively charged (quasi)particles described by Γ_e and Γ_h, respectively.

In conclusion, a slightly weaker assumption than (A5), accounting for the situation above, would be as follows:

(A5′) The "basic" charge carriers of a QH fluid are positively and/or negatively charged (quasi)particles. We assume that in the scaling limit the dynamics of positive-(quasi)particle-rich subfluids of a QH fluid is *independent* of the dynamics of negative-(quasi)particle-rich subfluids. The physically fundamental charge carriers of a QH fluid, electrons and/or holes, are *composites* of positive and/or negative "basic" (quasi)particles, respectively, or electrons and/or holes are *composites of both* positive *and* negative "basic" (quasi)particles.

Adopting assumption (A5′) instead of (A5), the classification problem of QH fluids (see Section 2) would be generalized according to: In the scaling limit, the quantum mechanical description of an (incompressible) QH fluid is universal. It is coded into a pair of odd, integral, Euclidean lattices, Γ_e *positive-* and Γ_h *negative*-definite, respectively, and an odd, primitive vector $\mathbf{Q} \in \Gamma^* = \Gamma_e^* \oplus \Gamma_h^*$.

For the reasons stated in Section 2 we do not study the resulting, slightly more general classification problem. We remark, however, that all hierarchy fluids which are of physical relevance in the region $0 < \sigma_{\mathrm{H}} < 1$ have been checked to belong to this more general classification program if they are not already contained in the one treated in this paper.

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